A Smoothing Technique For Objective Penalty Function In Inequality Constrained Optimization

Dr. Darpan Sood 1, Dr. Shaveta Kaushal

1Assistant Professor, Department of Mathematics, Sri Guru Teg Bahadur Khalsa College Anandpur Sahib Punjab, (India).140118

2Assistant Professor, Department of Economics, Sri Guru Teg Bahadur Khalsa College Anandpur Sahib Punjab, (India).140118

|  |
| --- |
| **Abstract:**  This manuscript provides a smoothing technique for objective penalty functions in inequality constrained optimization problems. A non- smooth penalty function is defined which is subjected to new smoothing technique to make it smooth. The error estimates for the original and the smoothed problem are discussed. A procedure is illustrated for the development of the solution of the inequality constrained optimization problem and is shown to be convergent under certain specified conditions.  **Keywords:** Smoothing, Penalty Function, Convergence,  Optimization Problem. |

**Section 1**

**Introduction**

The conventional inequality constrained optimization problem is defined in (1) as

(1) 

Where are derivable functions. The feasible set is represented by 

The penalty function method is an alternative technique in obtaining the solution of (1) in comparison to various other general methods. The penalty function method sets up a precedent whereas the problem is broken down into a concatenation of unconstrained optimization problem which are comparatively easy to solve. A common penalty function for the problem in (1) can be stated as

(2) 

Thus after invoking the above penalty function the optimization penalty problem (1) can be shaped as

(3) 

The penalty function defined in (2) is smooth but is not exact. By exactness we mean that for some  an optimal solution of (1) is also an optimal solution of (3) if. Initial work in the development of penalty function concept was done by Zangwill [1] in which the classical penalty function was defined as

(4) 

and the corresponding optimization problem for (1) was defined in the following manner

(5) 

Now the penalty function in (5) is bound to be exact but under certain conditions but is not smooth. The theory of exact penalty functions is attracting numerous researchers because of its naivety. Hans and Mangasrian [5] developed an exact penalty function. A globally convergent algorithm was provided by Rosenberg [3] for convex programming using the idea of exact penalty function. During the development of algorithms for obtaining the solution of penalty problems it was studied that the value of penalty parameter is to be increased gradually leading to non-derivability of penalty functions. [1, 3, 4, 5, 6]. Thus smoothing is must for the penalty functions to obtain a solution of constrained optimization problem using Newton methods or gradient based methods. Pinar and Zenios [5] presented a smoothed exact penalty function for convex constrained optimization problems involving all convex objective and constrained functions. The smoothing penalty function was first order differentiable. Yang et al. [7] in his paper discussed smoothing nonlinear penalty function for constrained optimization problem. Meng et al. [8] proposed smoothening of exact penalty function for inequality constrained optimization problem. Generally the exact penalty functions are not smooth and it is always desirable to have smoothing of penalty functions for solving the optimization problem. Thus to negate this problem various innovative strategies are being developed in the field of exact penalty function. One such technique is introduction of objective penalty parameter which has been thoroughly discussed in [9, 10, 11, 12, 13, 14] where in the penalty function is defined as

(6) where 

Let us assume as an optimal solution and as an optimal value of the objective function then SPFM (sequential penalty function method) can be contemplated and the solution at each step converges to the optimal solution. A general introduction to SPFM was given by Fiaccio and McCormick [10]. Meng et al. [14] also proposed and studied an objective function penalty method as

(7) 

which proved to be viable smooth penalty function for .

Meng and other co-authors [16] have also studied objective penalty function as

(8) 

The penalty function was shown to be smooth and exactness for objective penalty function was also proved in [16]. Various other forms of objective penalty function for solving the optimization problems can be defined by amalgamation of objective and constraint penalty. In the current article the smoothing of will be discussed.

The later part of the current research article is organized in the following manner. The section 2 comprises of certain theorems and their proofs for smoothening of penalty function. A methodical procedure for obtaining the solution of (1) is also being given in the paper in section 3. The convergence of the procedure is also obtained.

**Section 2**

**Smoothing Objective Penalty Function**

Let us define the function as follows

(9) 

This definition shows that does not belong to class of continuous functions on. The smoothening function is defined as

(10) 

On differentiating the function defined above we get

(11) 

The smoothing function defined above satisfies some wonderful properties in continuity and differentiability which are elucidated in the theorem proved below.

**Theorem 2.1**

For any  we have the following results

(i) is continuously differentiable on R

(ii) 

(iii) 

Proof:- (i)

The first part of the above theorem can be verified by showing the function to be continuous and at 0 and.

Continuity at 0

LHL = 

RHL= 

Thus LHL=RHL, Also 

Hence the function is continuous at 0

Continuity at 

LHL=

RHL=

Thus LHL= RHL, Also 

Hence we have shown the continuity of the function at 0 and .Now we can show that the function is first order continuously derivable. This can be achieved by showing the continuity of 

at both the points 0 and .

Now from equation (11) we have

(12) 

Continuity at 0

LHL=

RHL = 

Hence LHL= RHL, Also 

Hence the function is continuous at 0

Continuity at 

LHL=

RHL = 

Thus LHL = RHL, Also 

Hence the function is continuous at 0 and.

Hence the function is first order differentiable.

(ii) Consider both the functions and as defined above in equations (9) and (10) as





Now consider



Now when 

Define = 



Thus .

Also and 

Hence 

Thus 

(iii) Also from the above proof is quite evident that





Hence Proved

Now the problem defined in (1) can be modified as

(13) 

where  ,  .

The objective function in correspond to (11) is

(14) 

(15)  where . Thus depending on these penalty functions the two objective penalty problems can be represented as

(16) 

(17) 

The result proved in the next theorem gives us the relationship between the objective functions defined in (14) and (15).

**Theorem 2.2**

Prove that  is true for any.

Proof:-

From the result proved in the part (ii) of the above theorem we have the following inequality



Therefore as a consequence of the above inequality (18) can be rewritten as



Taking summation over for all  , the above inequality can be written as









**Theorem 2.3**

Consider a solution to the minimization problem for any arbitrary value M and is a sequence of positive numbers. Let the sequence of solution values  is having an accumulation point, and then is an optimal solution of.

**Definition 2.1**

Any  is termed as a feasible solution if 

**Theorem 2.4**

For any optimal solutions and of the problems defined in (16) and (17) , then



Proof:-

From the result proved in the theorem 2.1 we have the inequality









 

**Theorem 2.5**

Let us assume  as an optimal solution to (16) and also let be an optimal solution (17). Also further let is feasible to (1) and is considered to be feasible to (1), and then we have the following result



Proof: - As  is supposed to be feasible solution to (1) which implies that

(24) 

Moreover is assumed to be a feasible solution to (1) which again implies that

(25) 

Now using the inequality proved in theorem 2.2 we have the result





Using (24) and (25) in the above inequality we get



Hence Proved.

**Section 3**

**Algorithm for Smoothing of Objective Penalty Function**

The nonlinear optimization problem defined in (17) is an unconstrained optimization problem. The objective is to obtain a minimum solution of the problem which is a hardened procedure. To overcome this difficulty the problem is modified as. The algorithmic procedure for the evaluation of solution of the optimization problem is given below

Step1: Assume, , . Choose and satisfying the condition. Let and further suppose that. Find a sequence such that. Move to step 2.

Step 2: Solve the optimization problem. Let is a point which satisfies the problem.

Step3: Find the value. If  then let and move to step 5. If then move to step 4.

Step4: Let and go to step 5.

Step5: Stop the iterative procedure if and where and write is an solution to (1). If this is not the case then let and go to step 2 and repeat the procedure.

**Theorem 3.1**

Let be an open set. Further assume that is open and compact set. Suppose that the sequence generated by the algorithm defined above is and  is bounded. Then show that the sequence  is bounded and for any accumulation point of it there exist and such that

(26) 

Proof:

From the algorithm defined above it is quite evident that the sequence is increasing in nature and the sequence is decreasing in nature with the condition



Also



Which shows the convergence of both the sequences and. Assume that the sequence and. Now the result written above establishes the equality. Thus . From the statement we know is bounded and also, there must exist an such that



which leads to the boundedness of the sequence  .

Define the following sets for as







Assume that the sequence.

Now

(27) Where 

Now using (11) in the above inequality (27) we get

(28) 

Let us further suppose that for 

(29) 

Clearly the expression. Using (29) in (28) we get the following result

(30) 

Further put

(31) 

(32) 

(33) 

(34) 

Using the above inequalities in (29) and (30) we get

(35)  and

(36) 

As approaches, . Due to the continuity and differentiability of  we have .Thus from (34), (35) and (36) we have the desired result leading to the convergence of the algorithm.

**Section 4**

**Conclusion**

The present article is able provide a smoothing objective penalty function with certain results pertaining to error assessment of the smoothing objective penalty function are also proved in the article. An algorithmic design sequence based on the objective penalty function is also developed to obtain the solution of the constrained optimization problem. The global convergence of the algorithm is also proved in the article.

**References:-**

1. W.I. Zangwill (1967). Non Linear Programming via penalty function. Mgmt. Sci 13:334-358.
2. S.P.Han and O.L.Mangasrian(1979). Exact Penalty function in non-linear programming. Math.Prog. 17:251-269.
3. E. Rosenberg (1981). Globally Convergent algorithms for convex programming. Math.Oper.Res. 6:437-443.
4. M.C.Pinar and S.A. Zenios (1994). On smoothing exact penalty function for convex constraint optimization. SIAM J. Optimiz. 4:486-511.
5. S.A.Zenios , M.C,Pinar and R.S.Dembo (1993). A smooth penalty function algorithm for network structured problems. Eur. J. Oper. Res. 64:256-277.
6. X.Q. Yang, Z.Q.Meng, X.X.Huang and G.T.Y.Pong (2003) . Smoothing nonlinear penalty functions for constrained optimization. Numer. Funct. Anal. Optimiz. 24:351-364.
7. Z.Q.Meng, C.Y.Dang and X.Q. Yang (2006). On the smoothing of the square root exact penalty function for inequality constrained optimization Compu. Optimiz. Appls 35:375-398.
8. J.V.Burke(1991) . An exact penalization viewpoint of constrained optimization . SIAM J. Control Optimiz. 29:968-998.
9. A.V. Fiaccio and G.P. McComick (1968). Nonlinear Programming Sequential Unconstrained Minimization Techniques Wiley, New York.
10. R. Fletcher (1981). Practical Method of Optimization , Wiley-Interscience, New York.
11. R. Fletcher (1983) Penalty functions in Mathematical Programming: The state of the Art. Springer , Berlin , pp.87-114.
12. D. Mauricio and N. Maculan (2000) . A boolean penalty method for zero-one non linear programming . J. Global Optimiz. 16:343-354.
13. Z. Q. Meng, Q.Y.Hu and C.Y.Dang (2004) . An objective penalty function penalty function method for nonlinear programming, Appl Math Lett. 17: 683-689.
14. D.D. Morrison (1968) Optimization by least squares SIAM J. Numercal Analysis 5:83-88.