

An Economic Mathematical Fuzzy Model for Data Envelopment Analysis

Assem A. Tharwat¹, Basma E. El-Demerdash², Doaa W. Nada³

¹American University in the Emirates, UAE, assem.tharwat@aue.ae

²Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt, basma.ezzat@cu.edu.eg

³Newgiza University (NGU), Egypt, doaa.wafik@ngu.edu.eg

Abstract

Performance assessment is a central to the management process in any type of organization. In addition, making rational economical decisions to improve organizational performance is a daunting task, as any organization is typically a multi-faceted entity which rely on complex systems that use uncertain information. Data envelopment analysis (DEA) is a powerful quantitative tool that makes use of multiple inputs and outputs to obtain useful information about the performance and efficiency of an organization. In many real-life applications, observations are usually fuzzy in nature. Therefore, DEA efficiency measurement may be sensitive to such variations. The purpose of this study is to develop a unified economical fuzzy DEA model that handles variables of different natures (vague and deterministic) independently and can be adapted to both input- and output-oriented problems, whether it is constant/variable return to scale. To handle fuzzy variables specially the economic variables in the model, the α -cut approach was adopted. The model implementation is demonstrated through an illustrative case study. Managers will be able to use this model to identify and remedy underperformance, as well as to design regulations that aim to encourage efficiency and ensure that consumers benefit from the resulting efficiency gains.

Keywords: Data Envelopment Analysis; Benchmarking Uncertainty Model; Fuzzy Variables; Performance Assessment; Organization Improvement.

1. Introduction

For any organization, feedback and benchmarking are two essential components of economic performance evaluation and improvement. Track and field athletes, for instance, could not improve their performance if they do not know how fast they are running and what the record time is. Benchmarking is useful to identify top performers and

determine the gap with them so as to narrow any differences. For organizations, however, things get more complex because they represent clusters of sub-organizations, each with their own goals and motivations. In such a case, management's role consists in determining overall vision and goals for the entire organization which affects its economic and financial performance, while coordinating the efforts of each level of business in their achievement. The efficiency concepts are used for evaluating the effect of regulations and whether they play a constructive role. The basic definition of efficiency refers to the ability of an organization to produce the maximum output levels with a set of input levels [1,2].

Data Envelopment Analysis (DEA) is a non-parametric method based on linear programming that aims to derive the most efficient decision making Unit (DMU) that can be compared within a group of DMUs even with multiple inputs and outputs. Also, the term 'DEA' refers to the efficiency frontier that envelopes data. In addition, DEA is known to be a very powerful benchmarking technique. Although, many observations are important, reality problems could be fuzzy in nature. Moreover, the DEA model is sensitive to the different changes in variables. Therefore, an efficient DMU which is relatively efficient to other comparable DMUs could turn to be inefficient if such vagueness in variables, whether they are inputs, outputs or both, is present. In other words, if the collected data for a variable are not represented in the correct form, the resulting efficiencies will be erroneous and misleading because the efficiency scores are highly sensitive to the realized levels of inputs or outputs [3].

In recent years, good efforts have been made in the DEA models to address the vagueness in variables whether it is fuzzy input or fuzzy output. The applications of fuzzy DEA model are usually categorized into four approaches: tolerance approach, α -cut approach, fuzzy ranking approach, and possibility approach. The tolerance approach is considered the most popular fuzzy DEA model. Sengupta [4] was the first to express the fuzziness in the objective function and constraints and he developed a fuzzy mathematical programming model using the tolerance approach. The α -cut approach was proposed by Girod [5]. Its main idea is to convert the fuzzy DEA model to find the lower and upper bounds of the membership functions of the efficiency scores through a pair of parametric programs. Triantis and Girod [6] kept track of the beginning of fuzzy DEA model to measure technical efficiency by converting fuzzy input and fuzzy output variables into crisp variables using membership function. Kao and Liu [7] proposed a method to find the membership functions of fuzzy variables when data are fuzzy to measure relative efficiency using the fuzzy DEA model. He employed the α -cuts principle to convert a fuzzy DEA model to a family of crisp DEA models. Kao [8] proposed a DEA method for ranking the fuzzy relative efficiency scores where all input and output variables are fuzzy in nature. Saati et al. [9] proposed a fuzzy DEA model as a possibilistic

programming problem and converted it into an interval programming problem using α -cut approach. Entani et al. [10] developed a DEA approach for fuzzy input and output data by using α -level sets with an interval efficiency consisting of the efficiencies. Kao and Liu [11] developed a method to rank the fuzzy relative efficiency scores when all variables have fuzzy nature, i.e. the exact form of the membership functions are unknown.

Liu et al. [12] developed a modified fuzzy DEA model using an α -cut approach to handle fuzziness in input and output variables and incompleteness of information on weight indices in product design evaluation. Liu [13] proposed a fuzzy DEA model using an α -cut approach to find the relative efficiency scores when all variables were in fuzzy numbers. Zerafat et al. [14] developed a fuzzy DEA model based on an α -cut approach to retain the fuzziness of the model by maximizing the membership functions of inputs and outputs. Khoshfetrat and Daneshvar [15] proposed a modified fuzzy DEA model using the α -cut method, to convert the given fuzzy data to interval numbers, then compute the lower bounds of fuzzy inputs and outputs for each factor weight. Azadeh et al. [16] proposed a flexible approach composed of artificial neural network and fuzzy DEA for location optimization of solar plants. Zerafat et al. [17] introduced a DEA fuzzy model that can include some uncertainty information from the intervals within the α -cut approach. Kaleibar et al. [18] developed a fuzzy DEA model and used α -level set to transform fuzzy to crisp number for computing centralized resource allocation with a VRS. Hatami-Marbini et al. [19] proposed a fuzzy output-oriented DEA model that puts into consideration the vagueness of information to identify supplier performance which can be a flexible cross-efficiency evaluation methodology. Tharwat et al. [20] developed a fuzzy input-oriented DEA model that employs a combination of both fuzzy and deterministic output and/or input variables to be solved using the α -cut approach.

From the above literature survey, we reached that the available fuzzy output-oriented DEA model or the traditional DEA model consider all output and/or input variables as fuzzy in nature, although some might be deterministic. In addition, we found one paper only that tackles fuzzy input-oriented DEA model. Furthermore, there is no developed models that try to deal with the different orientation types either input or output and try to deal with the different return to scale types either constant or variable. So, in this study, we set to develop a unified fuzzy DEA model that adapts to both input- and output-oriented problems, whether CRS or VRS. In addition, this model handles variables of different natures independently (fuzzy and deterministic).

The upcoming section will discuss the methodology of the DEA models. The following (third) section includes the proposed unified fuzzy DEA

model. This is followed by a case study, after which the derived conclusions will be presented.

2. Data Envelopment Analysis General Mathematical Model

There are basically two main types of DEA models. The first is the constant return to scale (CRS) model, developed by Charnes et al. [21], where there is a direct proportion between changing outputs and changing inputs. The other is the variable return to scale (VRS) model, later developed by Banker et al. [22], where the changes in inputs may not result in direct proportional changes in outputs. The VRS model is one of the extensions of the CRS model where the efficient frontiers set is represented by a convex curve passing through all efficient DMUs. DEA models can also be classified as either input- or output-orientated. Input-oriented DEA models define the frontier by seeking the maximum possible proportional reduction in input usage, with output levels held constant, for each DMU. Output-orientated DEA models, on the other hand, seek the maximum proportional increase in output production, with input levels held fixed.

An input-/output-oriented CRS model is presented in model M-1 (for output Oriented - CRS model), and model M-2 (for input Oriented - CRS model).

- A basic Output Oriented - CRS model

$$\text{Max } W_p = \phi$$

s. t.

$$\begin{aligned} \sum_{i=1}^n \lambda_i y_{ik} &\geq \phi y_{pk}, \forall k = 1 \dots s \\ \sum_{i=1}^n \lambda_i x_{ij} &\leq x_{pj}, \forall j = 1 \dots m \\ \lambda_i &\geq 0, (i = 1, 2, \dots, n) \quad (M-1) \end{aligned}$$

- A basic Input Oriented - CRS model

$$\text{Min } W_p = \theta$$

s. t.

$$\begin{aligned} \sum_{i=1}^n \lambda_i x_{ij} &\leq \theta x_{pj}, \forall j = 1 \dots m \\ \sum_{i=1}^n \lambda_i y_{ik} &\geq y_{pk}, \forall k = 1 \dots s \\ \lambda_i &\geq 0, (i = 1, 2, \dots, n) \quad (M-2) \end{aligned}$$

where $k = 1$ to ' s ' (no. of outputs); $j = 1$ to ' m ' (no. of inputs); $i = 1$ to ' n ' (no. of DMUs); y_{ik} = amount of output k produced by DMU i ; x_{ij} = amount of input j utilized by DMU i ; λ_i = weight given to DMU i .

The most important extension of the original CRS models is given by additional constraint was introduced in models ($M-1$ and $M-2$): $\sum_{i=1}^n \lambda_i = 1$. This constraint enables variable returns to scale and provides that the reference set is formed as a convex combination of DMUs, which are in the set (those that have positive value for λ in the optimal solution). For output-oriented - VRS model (model $M-3$), the same model $M-1$ plus the inclusion of the convexity constraints and for input oriented - VRS model (model $M-4$), the same model $M-2$ plus the inclusion of the convexity constraints.

Based on the previous DEA definitions, it is noted that in the CCR models the output and input-oriented measures of efficiency are equal because there is a direct proportion between inputs and outputs variables. But in the BCC models, the output and input-oriented measures of efficiency scores are not equal for inefficient units because there are not necessarily proportional between changes in outputs and changes in the inputs [23].

3. Developed A Unified Fuzzy Data Envelopment Analysis Model

In real-life situations, the observed values are often imprecise or vague, such as in a manufacturing system, a production process or a service system, inputs and outputs are volatile and complex. It is difficult to measure them in an accurate way to obtain precise data. In addition, some of the variables available for measure efficiency will often be in the form of qualitative, linguistic data, e.g., "old" equipment and "good" service. Therefore, it was necessary to develop a fuzzy DEA model to deal with such cases. The DEA model can be adapted to model both input/output-oriented problems. Also, it can be used with CRS or VRS problems. So, in this section, we aim to develop a unified fuzzy DEA model that allows some input/output variables to be vague in nature while keeping other variables deterministic. Furthermore, the model can be adapted to model both input/output-oriented problems, whether CRS or VRS. To reach our main aim, we needed to go through four stages. The first stage requires defining a unified DEA model that can be adapted to model both input/output-oriented problems, whether CRS or VRS. The second stage involves specifying the α -cuts approach for the vague input and output variables. In the third stage, the equivalent crisp linear model of vague input variables for a unified DEA model is presented. Finally, in stage four, the equivalent crisp linear model of vague output variables for a unified DEA model is presented.

The following two remarks and two propositions to obtain the unified fuzzy DEA model for measuring the relative efficiency level for each

DMU that handle different natures of variable independently (fuzzy and deterministic) are presented.

Remark 1: Consider a DEA model that can be adapted to model both input/output-oriented problems, whether CRS or VRS. Then the unified DEA model based on (M-1 up to M-4) presented is as:

$$\begin{aligned}
 & \text{Max } W_p = \delta_{MT}\phi - (1 - \delta_{MT})\theta \\
 & \text{s. t.} \\
 & \sum_{i=1}^n \lambda_i x_{ij} \leq \delta_{MT}x_{pj} + (1 - \delta_{MT})\theta x_{pj} \quad , \forall j = 1 \dots m \\
 & \sum_{i=1}^n \lambda_i y_{ik} \geq \delta_{MT}\phi y_{pk} + (1 - \delta_{MT}) y_{pk} \quad , \forall k \\
 & \quad \quad \quad = 1 \dots s \quad \quad \quad (M - 5) \\
 & \delta_{RC} \left[\sum_{i=1}^n \lambda_i - 1 \right] = 0 \\
 & \lambda_i \geq 0, (i = 1, 2, \dots, n)
 \end{aligned}$$

where: δ_{MT} : the model type variable is defined as:

$$\delta_{MT} = \begin{cases} 1 & \text{if the model is Output oriented} \\ 0 & \text{if the model is Input oriented} \end{cases}$$

and δ_{RC} : The return to scale model type variable is defined as:

$$\delta_{RC} = \begin{cases} 1 & \text{if the model is variable reurn to scale} \\ 0 & \text{if the model is constant reurn to scale} \end{cases}$$

Consequently, there are three main different cases which are:

- $\delta_{MT} + \delta_{RC} = 2$ hence, the model is an output-oriented VRS DEA model presented in (M-3).
- $\delta_{MT} + \delta_{RC} = 1$
 - If $\delta_{MT} = 1$, hence, the model is an output-oriented CRS DEA model presented in (M-1).
 - If $\delta_{RC} = 1$, hence, the model is an input-oriented VRS DEA model presented in (M-4).
- $\delta_{MT} + \delta_{RC} = 0$, hence, the model is an input-oriented CRS DEA model presented in (M-2).

Remark 2: Suppose that some of the input and/or output observations are fuzzy variables, then the equivalent fuzzy DEA model for measuring the efficiency level of pth DMU for the model (M – 5) is as:

$$\begin{aligned}
 & \text{Max } W_p = \delta_{MT}\phi - (1 - \delta_{MT})\theta \\
 & \text{s. t.}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{i=1}^n \lambda_i x_{ij} &\leq \delta_{MT} x_{pj} + (1 - \delta_{MT}) \theta x_{pj} \quad , \forall j \in J_D \\
 \sum_{i=1}^n \lambda_i \tilde{x}_{ij} &\leq \delta_{MT} \tilde{x}_{pj} + (1 - \delta_{MT}) \theta \tilde{x}_{pj} \quad , \forall j \in J_F \\
 \sum_{i=1}^n \lambda_i y_{ik} &\geq \delta_{MT} \phi y_{pk} + (1 - \delta_{MT}) y_{pk} \quad , \forall k \\
 &\in K_D \quad (M-6) \\
 \sum_{i=1}^n \lambda_i \tilde{y}_{ik} &\geq \delta_{MT} \phi \tilde{y}_{pk} + (1 - \delta_{MT}) \tilde{y}_{pk} \quad , \forall k \in K_F \\
 \delta_{RC} \left[\sum_{i=1}^n \lambda_i - 1 \right] &= 0 \\
 \lambda_i &\geq 0, (i = 1, 2, \dots, n)
 \end{aligned}$$

where $\tilde{x}_{ij}$: fuzzy number for input j utilized by DMU i , $\tilde{y}_{ik}$: fuzzy number for output k produced by DMU i , J_D is the set of deterministic inputs, J_F is the set of fuzzy inputs, J is the set of all inputs, $J_D \cup J_F = J$ and K_D is the set of deterministic outputs, K_F is the set of fuzzy outputs, and K set of all outputs, where $K_D \cup K_F = K$.

The input-output observations are assumed fuzzy variables, then the deterministic inequalities convert to equivalent fuzzy inequalities based on the fuzzy theory [5]

Definition: a membership function for a fuzzy set A on the universe of discourse X is defined as $\mu_A: X \rightarrow [0,1]$, where each element of X is mapped to a value between 0 and 1. This value, called membership value or degree of membership, quantifies the grade of membership of the element in X to the fuzzy set A .

Membership functions allow us to graphically represent a fuzzy set. The x-axis represents the universe of discourse, whereas the y-axis represents the degrees of membership in the $[0,1]$ interval. Simple functions are used to build membership functions. There are different types of membership functions such as triangular, trapezoidal, etc.

Proposition 1: Assume that triangular membership function for fuzzy input variables ($\tilde{x}_{ij} \in J_F$), then the equivalent crisp linear model for the fuzzy DEA model using α -cut approach presented in the model (M-6) is as:

$$Max \tilde{W}_p = \delta_{MT} \phi - (1 - \delta_{MT}) \theta$$

s. t.

$$\begin{aligned}
 \sum_{i=1}^n \lambda_i x_{ij} &\leq \delta_{MT} x_{pj} + (1 - \delta_{MT}) \theta x_{pj} \quad , \forall j \in J_D \\
 \sum_{i=1}^n \lambda_i \tilde{x}_{ij} &\leq \delta_{MT} \tilde{x}_{pj} + (1 - \delta_{MT}) \theta \tilde{x}_{pj} \quad , \forall j \in J_F \\
 \tilde{x}_{ij} &\leq \alpha x_{ij}^M + (1 - \alpha) x_{ij}^U \quad , \forall j \in J_F, i = 1, 2, \dots, n \\
 \tilde{x}_{ij} &\geq \alpha x_{ij}^M + (1 - \alpha) x_{ij}^L \quad , \forall j \in J_F, i = 1, 2, \dots, n \\
 \sum_{i=1}^n \lambda_i y_{ik} &\geq \delta_{MT} \phi y_{pk} + (1 - \delta_{MT}) y_{pk} \quad , \forall k \\
 &\in K_D \quad (M-7) \\
 \sum_{i=1}^n \lambda_i \tilde{y}_{ik} &\geq \delta_{MT} \phi \tilde{y}_{pk} + (1 - \delta_{MT}) \tilde{y}_{pk} \quad , \forall k \in K_F \\
 \delta_{RC} \left[\sum_{i=1}^n \lambda_i - 1 \right] &= 0 \\
 \lambda_i &\geq 0, (i = 1, 2, \dots, n)
 \end{aligned}$$

Where α : α -cut level for fuzzy variables j , x_{ij}^L : the lower value of input fuzzy variable j utilized by DMU i , x_{ij}^M : median value of input fuzzy variable j utilized by DMU i , x_{ij}^U : the upper value of input fuzzy variable j utilized by DMU i .

Proof: For model (M-7) assumed that a triangular membership function for the input fuzzy numbers that are used for expressing fuzzy inputs as follows:

$$\mu_{\tilde{x}_{ij}} = \begin{cases} 0 & , x_{ij} \leq x_{ij}^L \\ \frac{x_{ij} - x_{ij}^L}{x_{ij}^M - x_{ij}^L} & , x_{ij}^L \leq x_{ij} \leq x_{ij}^M \\ \frac{x_{ij}^U - x_{ij}}{x_{ij}^U - x_{ij}^M} & , x_{ij}^M \leq x_{ij} \leq x_{ij}^U \\ 0 & , x_{ij} \geq x_{ij}^U \end{cases} \quad (1)$$

$$\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U), \quad 0 \leq x_{ij}^L \leq x_{ij}^M \leq x_{ij}^U \rightarrow \tilde{x}_{ij} \in [x_{ij}^L, x_{ij}^U] \quad (2)$$

To find α -cuts of $\tilde{x}_{ij}$ arithmetical operations on triangular fuzzy numbers are defined in eq. 1. With this operation, an interval that shows lower and upper bounds in different α -levels is found. Application of α -cut interval operations to fuzzy inputs are as follows:

$$\mu_{\tilde{x}_{ij}} \geq \alpha \begin{cases} \frac{x_{ij} - x_{ij}^L}{x_{ij}^M - x_{ij}^L} \geq \alpha \\ \frac{x_{ij}^U - x_{ij}}{x_{ij}^U - x_{ij}^M} \geq \alpha \end{cases} \quad (3)$$

$$\tilde{x}_{ij} \in [\alpha x_{ij}^M + (1 - \alpha) x_{ij}^L, \alpha x_{ij}^M + (1 - \alpha) x_{ij}^U] \quad (4)$$

Proposition 2: Assume that triangular membership function for fuzzy output variables ($\tilde{y}_{ik} \in K_F$), the equivalent crisp linear model for the fuzzy DEA model presented in the model (M-7) is as:

$$\begin{aligned} \text{Max } \tilde{W}_p &= \delta_{MT} \phi - (1 - \delta_{MT}) \theta \\ \text{s. t.} \end{aligned}$$

$$\sum_{i=1}^n \lambda_i x_{ij} \leq \delta_{MT} x_{pj} + (1 - \delta_{MT}) \theta x_{pj} \quad , \forall j \in J_D$$

$$\sum_{i=1}^n \lambda_i \tilde{x}_{ij} \leq \delta_{MT} \tilde{x}_{pj} + (1 - \delta_{MT}) \theta \tilde{x}_{pj} \quad , \forall j \in J_F$$

$$\tilde{x}_{ij} \leq \alpha x_{ij}^M + (1 - \alpha) x_{ij}^U \quad , \forall j \in J_F, i = 1, 2, \dots, n$$

$$\tilde{x}_{ij} \geq \alpha x_{ij}^M + (1 - \alpha) x_{ij}^L \quad , \forall j \in J_F, i = 1, 2, \dots, n$$

$$\sum_{i=1}^n \lambda_i y_{ik} \geq \delta_{MT} \phi y_{pk} + (1 - \delta_{MT}) y_{pk} \quad , \forall k \in K_D$$

(M-8)

$$\sum_{i=1}^n \lambda_i \tilde{y}_{ik} \geq \delta_{MT} \phi \tilde{y}_{pk} + (1 - \delta_{MT}) \tilde{y}_{pk} \quad , \forall k \in K_F$$

$$\tilde{y}_{ik} \leq \alpha y_{ik}^M + (1 - \alpha) y_{ik}^U \quad , \forall k \in K_F, i = 1, 2, \dots, n$$

$$\tilde{y}_{ik} \geq \alpha y_{ik}^M + (1 - \alpha) y_{ik}^L \quad , \forall k \in K_F, i = 1, 2, \dots, n$$

$$\delta_{RC} \left[\sum_{i=1}^n \lambda_i - 1 \right] = 0$$

$$\lambda_i \geq 0, (i = 1, 2, \dots, n)$$

Where α : α -cut level for fuzzy variables, y_{ik}^L : the lower value of fuzzy output variable k produced by DMU i , y_{ik}^M : median value of fuzzy output variable k produced by DMU i , y_{ik}^U : the upper value of the fuzzy output variable k produced by DMU i .

Proof: For model (M-8) assumed that a triangular membership function for the output fuzzy numbers that are used for expressing fuzzy outputs as follows:

$$\mu_{\tilde{y}_{ik}} = \begin{cases} 0 & , y_{ik} \leq y_{ik}^L \\ \frac{y_{ik} - y_{ik}^L}{y_{ik}^M - y_{ik}^L} & , y_{ik}^L \leq y_{ik} \leq y_{ik}^M \\ \frac{y_{ik}^U - y_{ik}}{y_{ik}^U - y_{ik}^M} & , y_{ik}^M \leq y_{ik} \leq y_{ik}^U \\ 0 & , y_{ik} \geq y_{ik}^U \end{cases} \quad (5)$$

$$\tilde{y}_{ik} = (y_{ik}^L, y_{ik}^M, y_{ik}^U), \quad 0 \leq y_{ik}^L \leq y_{ik}^M \leq y_{ik}^U \rightarrow \tilde{y}_{ik} \in [y_{ik}^L, y_{ik}^U] \quad (6)$$

To find α -cuts of $\tilde{y}_{ik}$, arithmetical operations on triangular fuzzy numbers are defined in eq. 5. With this operation, an interval that shows lower and upper bounds in different α -levels is found. Application of α -cut interval operations to fuzzy outputs are as follows:

$$\mu_{\tilde{y}_{ik}} \geq \alpha \begin{cases} \frac{y_{ik} - y_{ik}^L}{y_{ik}^M - y_{ik}^L} \geq \alpha \\ \frac{y_{ik}^U - y_{ik}}{y_{ik}^U - y_{ik}^M} \geq \alpha \end{cases} \quad (7)$$

$$\tilde{y}_{ik} \in [\alpha y_{ik}^M + (1 - \alpha)y_{ik}^L, \alpha y_{ik}^M + (1 - \alpha)y_{ik}^U] \quad (8)$$

4. Case Study

To illustrate the significance of our unified fuzzy DEA model (M-8) we present the following hypothetical case study which considers a factory for production smart TV with seven departments (maintenance, production, quality and control, inventory, procurement, marketing, and finance), three input variables and two output variables. Two of the selected input variables are deterministic (average number of working hours per department, and average monthly salaries per department) and based on the nature of the third input it considered as fuzzy variable (average employees' satisfaction level per department). In the other hand two output variables are selected, the first one represents the number of finished products/services per hour per department which is a deterministic one, and the second output represents the average organization's satisfaction level per department which is again a fuzzy variable. Both fuzzy variables are following triangular fuzzy numbers with minimum, average and maximum values for each department. The data that represents the deterministic variables and the parameters of fuzzy variables are provided in Tables 1,2. Furthermore, assume that the α -cut level for the problem is 0.5.

Table 1 Hypothetical data for the deterministic variables for the DMUs

Department	Inputs		Output
	Average Number of Working Hours (100)	Average Monthly Salaries (1000)	Finished Products/Services per Hour
Maintenance	6.11	4.36	21
Production	3.66	2.54	12
Quality and Control	1.44	0.48	14
Inventory	1.21	0.23	10
Procurement	2.75	1.40	10
Marketing	4.18	2.74	6
Finance	6.39	3.36	18

Table 2 Hypothetical data for the fuzzy variables for the DMUs

Department	Employees' Satisfaction Level	Organization Satisfaction Level
Maintenance	1.76 7.27 12.27	0.12 0.19 0.27
Production	3.85 4.65 5.53	0.00 0.10 0.24
Quality and Control	1.33 1.88 3.38	0.05 0.10 0.16
Inventory	0.78 1.48 2.06	0.00 0.06 0.16
Procurement	3.22 3.63 4.61	0.02 0.07 0.17
Marketing	4.30 6.13 8.03	0.00 0.06 0.15
Finance	4.40 8.00 10.68	0.06 0.17 0.30

To implement the proposed algorithm for solving the problem under investigation, two scenarios will be covered to determine the relative efficiency of the departments. In the first scenario, fuzzy output-oriented VRS DEA case as shown in the (M-8) when $MT + RC = 2$, an LP formulation for each department has to be provided in order to measure the relative efficiency. For the second scenario, fuzzy input-oriented VRS DEA case when $MT + RC = 1$ but $RC = 1$, an LP formulation for each department has to be provided in order to measure the relative efficiency. For each scenario, the GAMS programming language software used to solve the 7 models for each department independently. The relative efficiency level for each department is as shown in Table 3.

Table 3 Relative efficiency level for each DMU

Department	Fuzzy input-oriented DEA Model	Fuzzy Output-Oriented DEA Model
Maintenance	1	1
Production	0.36	0.44
Quality and Control	1	1
Inventory	1	1
Procurement	0.44	0.56
Marketing	0.29	0.13
Finance	0.80	0.66

Examining the results, we find that, the developed models showed promising comparable results. As mentioned above, the results are the same in the VRS DEA models either output- or input-oriented for efficient and inefficient DMUs but the efficiency scores for inefficient DMUs are not equal, because there are not necessarily proportional changes in the outputs and changes in the inputs. Table 3 showed that there are 3 efficient departments and 4 inefficient departments. For efficient departments, we advise comparing these departments with the corresponding departments in their competitors' factories to make sure these departments are efficient do not need any improvement because DEA is calculated relative efficiency, not efficiency in general.

For inefficient departments production, procurement, marketing, and finance have 0.4, 0.5, 0.21, and 0.73 average efficiency level respectively. We classify them from in three groups high, medium, and low inefficient. The interval of low inefficient group includes all inefficient department their efficiency level above 70%, medium inefficient group below 70% and above 30%, and high inefficient group below 30% efficiency level. From this perspective, marketing department classify as high inefficient, production and procurement departments classify as medium inefficient, and finance classify as high inefficient. High inefficient group need to take the largest share of the allocated budget for improvement. It is recommended to change in layout department and invested in employees training, this will be reflected directly on employee's satisfaction level, and indirectly way to optimize number of working hours to increase number of finish product/service per hour. In addition, invested in the marketing research to increased organization profit and of course the organization satisfaction level will be increased. This department need improvement on average by 80%.

For medium inefficient group, it is recommended to adding an additional economic product to increase the market share of the organization, new economic product did not rise the cost of the capital but increased the

net profit. Part of the allocated budget share for improvement can be used for the new product costs and another part will be invested in employees/labors training. The production and procurement need improvement in efficiency level on average by 60% and 50% respectively. Finally, for low inefficient group, need to take the lowest share of the allocated budget for improvement. It is recommended to change in layout department and invested in employees training to optimize the losing time in number of working hours and increase employee's satisfaction level. The finance department need to improve efficiency level by 27%. Moreover, it is recommended distributed as a quarterly profit for the employees/labors, this will be affected directly and indirectly to all parameters to increase the organization satisfaction level.

5. Conclusions

Because exact data may not always be available in real-life problem performance assessments due to the existence of uncertainty. In this study, a unified fuzzy DEA model was presented that is able to handle input and output variables with different nature (vague or deterministic) independently. The model can be adopted to model both input-oriented or output-oriented problems as well. Furthermore, it can be used with CRS or VRS problems. Input/output variables could be set to be deterministic, or fuzzy variables. Fuzzy variables are assumed with triangular membership functions. The idea is to transform the non-deterministic constraints to their equivalent deterministic constraint and solve the problem in deterministic domain. For this purpose, we used α -level approach to handle fuzzy variables.

The DEA efficiency measurement is sensitive to change in variables' nature. A DMU which is rated as efficient relative to other DMUs may turn inefficient if such uncertain variations are considered, or vice versa. In other words, if the collected data for a variable is not represented in the correct form, the resulting efficiencies will be erroneous and misleading because of the high sensitivity of the efficiency scores to the realized levels of inputs or outputs. Therefore, it is necessary to identify the nature of the variables from the beginning and apply the appropriate DEA model to achieve reliable results. Implementing the two models on the illustrative example resulted in: similar efficient DMUs and different the inefficient DMUs in terms of efficiency levels. This could not necessarily be interpreted by proportional changes in the outputs and changes in the inputs.

Bibliography

1. Bandyopadhyay, S. Effect of regulation on efficiency: evidence from Indian cement industry. *Central European Journal of Operations Research*, 2(2010): 153-170.
2. El-Demerdash, Basma E., Ihab A. El-Khodary, and Assem A. Tharwat. "A Stochastic Data Envelopment Analysis Model Considering Variation in Input and Output Variables." *International Journal of Data Envelopment Analysis and *Operations Research** 2(1) (2016): 1-6.
3. El-Demerdash, B. E., El-Khodary, I. A., & Tharwat, A. A. Developing a stochastic input oriented data envelopment analysis (SIODEA) model. *International Journal of Advanced Computer Science and Applications*, 4(4) (2013).
4. Sengupta, J.K. A fuzzy systems approach in data envelopment analysis. *Computers & Mathematics with Applications*, 8(1992): 259-266.
5. Girod, O.A. Measuring technical efficiency in a fuzzy environment (Doctoral dissertation, Virginia Polytechnic Institute and State University), 1996.
6. Triantis, K., and Girod, O. A mathematical programming approach for measuring technical efficiency in a fuzzy environment. *Journal of Productivity Analysis*, 1(1998): 85-102.
7. Kao, C., and Liu, S.T. Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems*, 3(2000): 427-437.
8. Kao, C. A mathematical programming approach to fuzzy efficiency ranking. In *Fuzzy Systems, 2001. The 10th IEEE International Conference*, 1(2001): 216-219.
9. Saati, S.M., Memariani, A., and Jahanshahloo, G.R. Efficiency analysis and ranking of DMUs with fuzzy data. *Fuzzy Optimization and Decision Making*, 3(2002): 255-267.
10. Entani, T., Maeda, Y., and Tanaka, H. Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, 1(2002): 32-45.
11. Kao, C., and Liu, S.T. A mathematical programming approach to fuzzy efficiency ranking. *International Journal of Production Economics*, 2(2003): 145-154.
12. Liu, Y.P., Gao, X.L., and Shen, Z.Y. Product design schemes evaluation based on fuzzy DEA. *Computer Integrated Manufacturing Systems-Beijing*, 11(2007): 2099-2104.
13. Liu, S.T. A fuzzy DEA/AR approach to the selection of flexible manufacturing systems. *Computers & Industrial Engineering*, 1(2008): 66-76.
14. Zerafat Angiz, L.M., Emrouznejad, A., and Mustafa, A. Fuzzy assessment of performance of a decision making units using DEA: A non-radial approach. *Expert Systems with Applications*, 7(2010): 5153-5157.
15. Khoshfetrat, S., and Daneshvar, S. Improving weak efficiency frontiers in the fuzzy data envelopment analysis models. *Applied Mathematical Modelling*, 1(2011): 339-345.
16. Azadeh, A., Sheikhalishahi, M., and Asadzadeh, S.M. A flexible neural network-fuzzy data envelopment analysis approach for location

- optimization of solar plants with uncertainty and complexity. *Renewable Energy*. 12(2011): 3394-3401.
17. Zerafat Angiz, L.M., Emrouznejad, A., and Mustafa, A. Fuzzy data envelopment analysis: A discrete approach. *Expert Systems with Applications*, 3(2012): 2263-2269.
18. Kaleibar, M., Ashrafi, A., & Daneshvar, S. Modified Centralized Resource Allocation Model with Variable Return to Scale for Fuzzy Data. *Recent Developments in Data Envelopment Analysis and its Applications*. (2014): 292-304.
19. Hatami-Marbini, A., Agrell, P. J., Tavana, M., & Khoshnevis, P. A flexible cross-efficiency fuzzy data envelopment analysis model for sustainable sourcin. *Journal of Cleaner Production*. 142 (2017): 2761—2779.
20. Tharwat, A. A., El-Demerdash, B. E., & El-Khodary, I. A. Developing an Input Oriented Data Envelopment Analysis Model with Fuzzy Uncertainty in Variables. *Creative Business and Social Innovations for a Sustainable Future*, Springer. (2019): 117-122.
21. Charnes A., Cooper W., & Rhodes E. Measuring the efficiency of efficiency of decision-making units. *European Journal of Operational Research*. 2(6) (1978): 429-444.
22. Banker R., Charnes A., & Cooper W. Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science*. 30 (1984): 1078-1092.
23. Martić, M., Novaković, M., & Baggia, A. Data envelopment analysis-basic models and their Utilization. *Organizacija*, 42(2) (2009): 37-43.