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Clique And Path Non Split Domination Of G_n- Graph

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Abstract:

An undirected graph G_n defined on a finite subset of natural numbers, is an undirected graph whose vertex set $V = \{x \in N: \gcd(x, n) \neq 1, x < n\}$ and $x, y \in V$ are adjacent if and only if $\gcd(x, y) > 1$. In this paper, Clique domination and Path non-split domination is discussed and the some results are presented at various values of n of G_n.

Keywords: Undirected graph, Domination, Clique Domination, Path non split domination.

Mathematics Subject Classification: 05C25, 05C69

1. INTRODUCTION

"Domination is the most popular concept in graph theory and it was first introduced by Berge [1] and Ore [10]. The dominating set $D \subset V$ of a graph G (V, E) is defined as every vertex v in $V - D$ is adjacent to some vertex in D and the minimum cardinality of a dominating set is the domination number of G, denoted by $\gamma(G)$. Total domination was introduced by Cockayne et al. [3,4,5] and some concepts of this domination, found in Haynes et al. [7]. The relation between domination number and total domination number of a graph without isolated vertices was given in Bollobas [2]." Domination parameters of involutory cayley graph was discussed by Prameela et al.[11,12]. Shanmugha Priya et al. [13,14] developed an Involutory Addition Cayley graph in 2020 and discussed the structural properties and Strong domination of Unitary Addition Cayley graph.

In this paper, some results on clique and path non split dominating sets of an undirected graph on a finite subset of natural numbers are presented and the respective domination numbers are obtained.

2. UNDIRECTED G_n GRAPH AND ITS PROPERTIES

Cayley in 1978 constructed the digraph of a group, thus paving way for many more emerging graphs for semi groups such as divisibility graphs, power graphs, annihilator graphs and so on. The Involutory Cayley graph was introduced by Venkata Anusha et al. [15] and studied some basic properties.

Chakrabarty [8] developed the concept of an undirected graph on a finite subset of natural mber and it is denoted by G_n.

“**Definition:** Let $n \in N$ be a composite number. An undirected G_n graph is defined as a graph whose vertex set $V = \{x \in N: \gcd(x, n) \neq 1, x < n\}$ where $x, y \in \mathrm{V}$ are adjacent if and only if $\gcd(x, y) > 1$. “

Some of the properties of undirected G_n graph given by Ivy [8] are

Lemma 2.1 [8]: The graph G_n is disconnected if and only if $n = 2p$, where p is an odd prime. Moreover, the components of $G_{2p}$are K_{p-1} and K_1.

Lemma 2.2 [8]: The graph G_n is complete if and only if $n = p^m$, where p is a prime.

3. CLIQUE DOMINATION IN GRAPH G_n

The domination clique was introduced by Cozzens and Kelleher [6] and established a sufficient condition for a graph to have a dominating clique in terms of forbidden subgraphs. In this section the concept of clique domination of a graph G_n are studied and clique domination number of G_n are obtained for different values of n.

Definition: A dominating set D of vertices in a graph $G = (V, E)$ is a dominating clique if the induced sub graph $\langle D \rangle$ is complete. The clique domination number of a graph G is written as $\gamma_{\mathrm{cl}}(G)$ and it is the minimum cardinality of a dominating clique of G.

Theorem 3.1: For a graph G_n, the clique domination number $\gamma_{\mathrm{ns}}(G_n) = 1$, if $n = p^m$, where p is prime.

Proof: Consider the graph G_n, where $n = p^m$, and p is prime. Then the vertex set of graph G_n consisting of all multiples of p which are less than n. That means $V = \{p, 2p, 3p, \dots\dots\dots, (p^{m-1} - 1)p\}$ and $|V| = p^{m-1} - 1$.
By the definition of graph G_n, it forms a complete graph and degree of each vertex is $|V| - 1$, which is maximum. Therefore every vertex $v_i = i$ in V is adjacent to all $v_j = j$ where $i \neq j$ of G_n as $\gcd(i, j) > 1$.
Define $D = \{p\}$. It implies D forms a dominating set to G_n with minimum cardinality 1. Now the induced subgraph $\langle D \rangle$ of G_n has only one vertex and it is complete.
Therefore D becomes a dominating clique of G_n and hence, $\gamma_{\mathrm{cl}}(G_n) = 1$.

Theorem 3.2: For a graph G_n, if $n = p_1.p_2$ where p_1 and p_2 are primes, the clique domination number $\gamma_{\mathrm{ns}}(G_n) = 2$.

Proof: Consider the graph G_n, with $n = p_1.p_2$ where p_1 and p_2 are primes.

Case 1: Let $p_1 = 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.
Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and
$S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.
If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = p_2 - 1$, which is of maximum degree.
If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.
Define a set
$D = \{u, v: u = kp_1, v = lp_2\ where\ k, l\ are\ even\}$. Since the vertices of D dominates all other vertices in a graph G_n, it follows that D forms a dominating set which is of minimum cardinality.
The vertices in a dominating set are adjacent and implies $\langle D \rangle$ is complete in G_n. Therefore D becomes a dominating clique of G_n and hence, $\gamma_{\mathrm{cl}}(G_n) = 2$.

Case 2: Let $p_1 > 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.
Now divide the vertex set V into two subsets as $S_1 = \{u: u = kp_1 < n, where\ k \in N\}$ and
$S_2 = \{v: v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = \frac{2p_2+p_1-5}{2} + \left\lfloor\frac{p_1-1}{3}\right\rfloor - \left\lfloor\frac{p_2-1}{6}\right\rfloor$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Define a set

$D = \{u, v: u = kp_1, v = lp_2 \ where \ k, l \ are \ even\}$. Since the vertices of D dominates all other vertices in a graph G_n, it follows that D forms a dominating set which is of minimum cardinality. The vertices in a dominating set are adjacent and implies $\langle D\rangle$ is complete in G_n. Therefore D becomes a dominating clique of G_n and hence, $\gamma_{\text{cl}}(G_n) = 2$.

Theorem 3.3: For a graph G_n, if $n = {p_1}^{a_1}.{p_2}^{a_2} \dots \dots . . {p_m}^{a_m}$ where $p_1, p_2, \dots . p_m$ are primes and $a_1, a_2, \dots ., a_m$ are natural numbers, the clique domination number $\gamma_{\text{ns}}(G_n) = 1$.

Proof: Consider the graph G_n, with

$n = {p_1}^{a_1}.{p_2}^{a_2} \dots \dots . . {p_m}^{a_m}$ where $p_1, p_2, \dots . p_m$ are primes and $a_1, a_2, \dots ., a_m \in N$.

Then for the graph G_n, the vertex set V contains all the multiples of $p_1, p_2, \dots . ., p_m$ which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into $m-$ subsets as $S_1, \ S_2, \dots \dots \dots , \ S_m$ where $S_i = \{v_i: v_i = k_i p_i < n, where \ k_i \in N\}$ for $i = 1 \ to \ m$. Then vertex $p_1 . p_2 \dots . . p_m$ is a common vertex of all the $S_i{}'s$ and it is adjacent to every vertex in the graph.

A set D$= \{p_1 . p_2 \dots . . p_m\}$ forms a dominating set for G_n and the induced subgraph $\langle D\rangle$ is complete in G_n. Therefore D becomes a dominating clique of G_n and hence, $\gamma_{\text{cl}}(G_n) = 1$.

4. PATH NON-SPLIT DOMINATION IN A GRAPH G_n

Kulli and Nandargi [9] introduced the notion of path nonsplit domination in graph and proved that a trivial graph has no a path nonsplit dominating set. In this section results on path non-split domination for an undirected graph G_n for different values of n are presented.

Definition: Let $G(V, E)$ be a graph. A dominating set D of G is a path non-split dominating set of G if the induced subgraph $\langle V - D\rangle$ is a path in G.

The path non-split domination number $\gamma_{\text{pns}}(G)$ of G is the minimum cardinality of a path non-split dominating set of G.

Theorem 4.1: For a graph G_n, the path non split domination number $\gamma_{pns}(G_n) = 1$, if $n = p^m$, where p is prime.

Proof: Consider the graph G_n, where $n = p^m$, and p is prime. Then the vertex set of graph G_n consisting of all multiples of p which are less than n. That means $V = \{p, 2p, 3p, \dots \dots \dots, (p^{m-1} - 1)p\}$ and
$|V| = p^{m-1} - 1$.

By the definition of graph G_n, it forms a complete graph and degree of each vertex is $|V| - 1$, which is maximum. Therefore every vertex $v_i = i$ in V is adjacent to all $v_j = j$ where $i \neq j$ of G_n as $\gcd(i, j) > 1$.

Then $D = \{p\}$ forms a dominating set, and the induced subgraph $\langle \mathrm{D} \rangle$ has only one vertex and it is P_1. Hence the path non split domination number of G_n, $\gamma_{pns}(G_n) = 1$, if $n = p^m$, where m is prime.

Theorem 4.2: For a graph G_n, if $n = p_1 . p_2$ where p_1 and p_2 are primes, the path non split domination number $\gamma_{\mathrm{pns}}(G_n) = 2$.

Proof: Consider the graph G_n, with $n = p_1 . p_2$ where p_1 and p_2 are primes.

Case 1: Let $p_1 = 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u : u = kp_1 < n, where\ k \in N\}$ and
$S_2 = \{v : v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = p_2 - 1$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Then $D = \{u, v\}$ forms a dominating set and the induced subgraph $\langle \mathrm{D} \rangle$ is a path with 2 vertices. Therefore D is path dominating set of G_n.

Hence the path non split domination number of G_n, $\gamma_{pns}(G_n) = 2$.

Case 2: Let $p_1 > 3$. Then for the graph G_n, the vertex set V contains all p_1 −multiples and p_2 − multiples which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into two subsets as $S_1 = \{u : u = kp_1 < n, where\ k \in N\}$ and
$S_2 = \{v : v = lp_2 < n, where\ l \in N\}$.

If k is even, then the vertex u in S_1 is adjacent to all vertices of S_1 and also adjacent to all even vertices of S_2 and $degree(u) = \frac{2p_2+p_1-5}{2} + \left\lfloor\frac{p_1-1}{3}\right\rfloor - \left\lfloor\frac{p_2-1}{6}\right\rfloor$, which is of maximum degree.

If l is even, then the vertex v in S_2 is an even number and $gcd(u, v) = 2$.

Then $D = \{u, v\}$ forms a dominating set and the induced subgraph $\langle \mathrm{D} \rangle$ is a path with 2 vertices. Therefore D is path dominating set of G_n.

Hence the path non split domination number of G_n, $\gamma_{pns}(G_n) = 2$.

Theorem 4.3: For a graph G_n, if $n = {p_1}^{a_1}.{p_2}^{a_2} \dots \dots .. {p_m}^{a_m}$ where $p_1, p_2, \dots.\ p_m$ are primes and $a_1, a_2, \dots., a_m$ are natural numbers, the path non split domination number $\gamma_{pns}(G_n) = 1$.

Proof: Consider the graph G_n, with $n = {p_1}^{a_1}.{p_2}^{a_2} \dots \dots .. {p_m}^{a_m}$ where $p_1, p_2, \dots.\ p_m$ are primes and $a_1, a_2, \dots., a_m \in N$.

Then for the graph G_n, the vertex set V contains all the multiples of $p_1, p_2, \dots .., p_m$ which are less than n and $|V| = n - \emptyset(n) - 1$.

Now divide the vertex set V into $m-$ subsets as $S_1,\ S_2, \dots \dots \dots,\ S_m$ where

$S_i = \{v_i : v_i = k_i p_i < n, where\ k_i \in N\}$ for $i = 1\ to\ m$.

Then vertex $p_1.p_2 \dots .. p_m$ is a common vertex of all the ${S_i}'s$ and it is adjacent to every vertex in the graph.

Therefore a set D= $\{p_1.p_2 \dots .. p_m,\}$ forms a dominating set and the induced subgraph $\langle \mathrm{D} \rangle$ has only one vertex and it is P_1.

Hence the path non split domination number of G_n, $\gamma_{pns}(G_n) = 1$.

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