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A New Class Of Analytic Univalent Functions And Its Fekete Szego Coefficient Inequality

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ABSTRACT:
We have introduced subclasses of analytic functions and have obtained sharp upper bounds of the Fekete Szego functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

KEYWORDS: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

Introduction : Let us denote by the symbol $\mathcal{A}$, the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit disc of the complex plane given by $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of those analytic functions of the above defined form (1.1), which are univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([3]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [15] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö [6] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, \text{if} \mu \geq 1. \end{cases} \tag{1.2}$$

This inequality (1.2) played a vital role in obtaining estimates of higher coefficients for some sub classes $\boldsymbol{\mathcal{S}}$ ([1], [2], [4], [5], [7]-[14]).

Let us define some subclasses of $\boldsymbol{\mathcal{S}}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$\operatorname{Re}\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$\operatorname{Re}\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*(A, B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\}$$

(1.6)

$$\mathcal{K}(A, B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\}$$

(1.7)

It is obvious that $S^*(A, B)$ is a subclass of S^* and $\mathcal{K}$ (A, B) is a subclass of $\mathcal{K}$.

Several authors studied and introduced various classes and subclasses of univalent analytic functions and established Fekete Szego inequality for the same. ([22]-[67])

We introduce here a new class as

$$D^*(f) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{2zf'(z)}{f(z) + f\left(\frac{z}{2}\right) + f\left(\frac{z}{2^2}\right) + f\left(\frac{z}{2^3}\right) + ----} \prec \frac{1+z}{1-z}; z \in \mathbb{E}\right\}$$

and will establish its coefficient inequality.

We will deal with some subclasses of $D^*(f)$ defined as follows in the next paper:

$$D^*(f,A,B)=\left\{f(z)\in\boldsymbol{\mathcal{A}};\frac{2zf'(z)}{f(z)+f\left(\frac{z}{2}\right)+f\left(\frac{z}{2^2}\right)+f\left(\frac{z}{2^3}\right)+----}\prec\frac{1+Az}{1+Bz};z\in\mathbb{E}\right\}$$

$$D^*(f,\delta)=\left\{f(z)\in\boldsymbol{\mathcal{A}};\frac{2zf'(z)}{f(z)+f\left(\frac{z}{2}\right)+f\left(\frac{z}{2^2}\right)+f\left(\frac{z}{2^3}\right)\pm---}\prec\left(\frac{1+z}{1-z}\right)^{\delta};z\in\mathbb{E}\right\}$$

$$D^*(f,A,B,\delta)=\left\{f(z)\in\boldsymbol{\mathcal{A}};\frac{2zf'(z)}{f(z)+f\left(\frac{z}{2}\right)+f\left(\frac{z}{2^2}\right)+f\left(\frac{z}{2^3}\right)+----}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta};z\in\mathbb{E}\right\}$$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to $F(z)$ in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$; $z\in\mathbb{E}$ and we write $f(z)\prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z)=\sum_{n=1}^{\infty}d_n z^n$, $w(0)=0$, $|w(z)|<1$.
(1.10)

It is known that $|d_1|\le 1, |d_2|\le 1-|d_1|^2$.
(1.11)

PRELIMINARY LEMMAS: For $0<c<1$, we write $w(z)=\left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)}=1+2cz+2z^2+\cdots.$$
(2.1)

MAIN RESULTS

THEOREM 3.1: Let $f(z)\in D^*(f)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{70}{3}-36\mu\text{: if } \mu \geq \frac{14}{27} & (3.1) \\ \frac{14}{3}\text{: if } \frac{14}{3} \leq \mu \leq \frac{7}{9} & (3.2) \\ 36\mu-\frac{70}{3}\text{: if } \mu \geq \frac{7}{9} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $D^*(f)$, we have

$$\frac{2zf'(z)}{f(z)+f\left(\frac{z}{2}\right)+f\left(\frac{z}{2^2}\right)+f\left(\frac{z}{2^3}\right)+----} \prec \frac{1+z}{1-z} \tag{3.4}$$

Expanding the series (3.4), we get

$\{1+a_2z+a_3z^2+---\}=(1+\frac{2}{3}a_2z+\frac{4}{7}a_3z^2+---)(1+2c_1z+2(c_2+{c_1}^2)z^2+---)$.(3.5)

Identifying terms in (3.5), we get

$$a_2=6c_1 \tag{3.6}$$

$$a_3=\frac{14}{3}(c_2+\ 5{c_1}^2). \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$a_3-\mu a_2^2=\frac{14}{3}c_2+\left[\frac{70}{3}-36\mu\right]c_1^2. \tag{3.8}$$

Taking absolute value, (3.8) can be rewritten as

$$\left|a_3-\mu a_2^2\right| \leq \frac{14}{3}\left|c_2\right|+\left|\frac{70}{3}-36\mu\right|\left|c_1^2\right| \tag{3.9}$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{14}{3}(1 - |c_1|^2) + \left|\frac{70}{3} - 36\mu\right| |c_1^2|$$
$$= \frac{14}{3} + \left[\left|\frac{70}{3} - 36\mu\right| - \frac{14}{3}\right] |c_1|^2. \qquad (3.10)$$

Case I: $\mu \leq \frac{35}{54}$.

(3.10) can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \leq \frac{14}{3} + \left[\frac{56}{3} - 36\mu\right] |{c_1}^2|. \qquad (3.11)$$

Subcase I (a): $\mu \leq \frac{14}{27}$.

Using (1.9), (3.11) becomes

$$\left|a_3 - \mu a_2^2\right| \leq \frac{70}{3} - 36\mu \qquad (3.12)$$

Subcase I (b): $\mu \geq \frac{14}{27}$.

We obtain from (3.11)

$$\left|a_3 - \mu a_2^2\right| \leq \frac{14}{3}. \qquad (3.13)$$

Case II: $\mu \geq \frac{35}{54}$

Preceding as in case I, we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{14}{3} + [36\mu - 28]|{c_1}^2|. \qquad (3.14)$$

Subcase II (a): $\mu \leq \frac{7}{9}$

(3.14) takes the form

$$|a_3 - \mu a_2^2 \leq \frac{14}{3} \tag{3.15}$$

Combining subcase I (b) and subcase II (a), we obtain

$$\frac{14}{3} : if \ \frac{14}{3} \leq \mu \leq \frac{7}{9} \tag{3.16}$$

Subcase II (b): $\mu \ \geq \frac{7}{9}$

Preceding as in subcase I (a), we get

$$\left|a_3 - \mu a_2^2\right| \leq 36\mu - \frac{70}{3} \tag{3.17}$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

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