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Another Linearly Distributed Periodic Subclass Of Class Of Starlike Functions And Its Coefficient Inequality

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ABSTRACT:
In the present paper, we have established coefficient inequality for a linearly distributed periodic subclass of class of starlike analytic functions i.e. sharp upper bounds of the Fekete–Szegö functional $|a_3 - \mu a_2^2|$ for the analytic functions of the type $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this subclass.

KEYWORDS: Linearly distributed subclass, Univalent functions, Coefficient inequality, Starlike functions, Analytic functions and bounded functions.

Introduction

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction : We denote the class of functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$
(1.1)

which are analytic in the unit disc given by $\mathbb{E} = \{z: |z| < 1|\}$, by the symbol $\mathcal{A}$, We denote the class of functions of the type (1.1), which are analytic as well as univalent in $\mathbb{E}$, by the symbol $\mathcal{S}$,

In 1916, Bieber Bach ([3]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [14] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was expected to try to find some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$,Fekete and Szegö[4] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{bmatrix} 3-4\mu & \text{if } \mu \leq 0; \\ 1 & , \text{if } 0 \leq \mu \leq 1; \\ 4\mu-3, & \text{if} \mu \geq 1. \end{bmatrix}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\boldsymbol{S}$([3], [9]).
Let us define some subclasses of $\boldsymbol{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$\operatorname{Re}\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$\operatorname{Re}\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$\operatorname{Re}\left(\frac{zf\prime(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by C and was introduced by Duran [6] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$$

(1.6)

$$\mathcal{K}(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{\left(zf'(z)\right)\prime}{f\prime(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$$

(1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}$ (A,B) is a subclass of $\mathcal{K}$.

Several authors introduced various classes and subclasses of univalent functions and established coefficient inequalities for these classes ([1], [2], [4]-[8], [10]-[54]).

In our previous paper, we introduced the class LDS^* and established coefficient inequality ([4]) for the same.

Now, we introduce linearly distributed periodic subclass as

$$\left\{f(z) \in \boldsymbol{\mathcal{A}}; \left(\frac{\alpha z f'(z)}{f(\alpha z)}\right) \prec \left\{\frac{1 + Aw(z)}{1 + Bw(z)}\right\}; z \in \mathbb{E}, -1 \leq B < A \leq 1\right\}$$

and we will denote this class as $LDS^*(A, B)$.

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to $F(z)$ in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$

(1.10)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$.

(1.11)

2. **PRELIMINARY LEMMAS:** For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots.$

(2.1)

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in LDS^*(A, B)$, then

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3 - \mu a_2^2\right|$$
$$\leq \begin{cases} (\alpha A - 2B) - \dfrac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu, & \text{if } \mu \leq \dfrac{(2-\alpha)(\alpha A - 2B + \alpha - 2)}{(A-B)(3-\alpha^2)}; \quad (3.1) \\ (2-\alpha), \quad \text{if } \dfrac{(2-\alpha)(\alpha A - 2B + \alpha - 2)}{(A-B)(3-\alpha^2)} \leq \mu \leq \dfrac{(2-\alpha)(\alpha A - 2B + \alpha - 2)}{(A-B)(3-\alpha^2)}; & (3.2) \\ \dfrac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu - (\alpha A - 2B), & \text{if } \mu \geq \dfrac{(2-\alpha)(\alpha A - 2B + \alpha - 2)}{(A-B)(3-\alpha^2)}; \quad (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $LDS^*(A, B)$, we have

$\left(\frac{\alpha z f'(z)}{f(\alpha z)}\right) \prec \left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}; w(z) \in \mathcal{U}.$

(3.4)

Expanding the series (3.4) and identifying terms, we get

$$\mathrm{a}_2 = \frac{\mathrm{A-B}}{2-\alpha}\mathrm{c}_1$$
(3.5)

$$a_3 = \frac{A-B}{3-\alpha^2}c_2 + \frac{(A-B)(\alpha A-2B)}{(3-\alpha^2)(2-\alpha)}{c_1}^2.$$
(3.6)

From (3.5) and (3.6), we obtain

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}(a_3-\mu a_2^2) = (2-\alpha)c_2 + \left[(\alpha A-2B) - \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu\right]c_1^2. \quad (3.7)$$

Taking absolute value and using Triangular inequality, (3.7) can be rewritten as

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = (2-\alpha)|c_2| + \left|(\alpha A-2B) - \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu\right|\left|c_1^2\right|. \quad (3.8)$$

Using (1.9) in (3.8), Simple calculations yield

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = (2-\alpha) + \left\{\left|(\alpha A-2B) - \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu\right| - (2-\alpha)\right\}\left|c_1^2\right|. \quad (3.9)$$

Case I: $\mu \leq \frac{(2-\alpha)(\alpha A-2B)}{(A-B)(3-\alpha^2)}$.

In this case, (3.9) can be rewritten as

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = (2-\alpha) + \left\{(\alpha A-2B-2+\alpha) - \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu\right\}\left|c_1^2\right|. \quad (3.10)$$

Subcase I (a): $\mu \leq \frac{(2-\alpha)(\alpha A-2B+\alpha-2)}{(A-B)(3-\alpha^2)}$.

Using (1.9), (3.10) becomes

$$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = (\alpha A-2B) - \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu.$$
(3.11)

Subcase I (b): $\mu \geq \frac{(2-\alpha)(\alpha A-2B+\alpha-2)}{(A-B)(3-\alpha^2)}$

We obtain from (3.11)

$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = 2-\alpha.$
(3.12)

Case II: $\mu \geq \frac{(2-\alpha)(\alpha A-2B)}{(A-B)(3-\alpha^2)}$

Preceding as in case I, we get

$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = (2-\alpha)+\left\{\frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu-(\alpha A-2B+2-\alpha)\right\}\left|c_1^2\right|.$ (3.13)

Subcase II (a): $\mu \leq \frac{(2-\alpha)(\alpha A-2B-\alpha+2)}{(A-B)(3-\alpha^2)}$

(3.13) takes the form

$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = 2-\alpha.$
(3.14)

Combining subcase I (b) and subcase II (a), we obtain

$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = 2-\alpha,\ if\ \frac{(2-\alpha)(\alpha A-2B+\alpha-2)}{(A-B)(3-\alpha^2)} \leq \mu \leq \frac{(2-\alpha)(\alpha A-2B-\alpha+2)}{(A-B)(3-\alpha^2)}$ (3.15)

Subcase II (b): $\mu \geq \frac{(2-\alpha)(\alpha A-2B-\alpha+2)}{(A-B)(3-\alpha^2)}$

Preceding as in subcase I (a), we get

$\frac{(3-\alpha^2)(2-\alpha)}{A-B}\left|a_3-\mu a_2^2\right| = \frac{(A-B)(3-\alpha^2)}{(2-\alpha)}\mu-(\alpha A-2B).$
(3.16)

Combining (3.11), (3.15) and (3.16), the theorem is proved.

Corollary 3.2: Putting $\alpha = 1$, in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu, if \mu \leq \dfrac{1}{2}; \\ 1 if \dfrac{1}{2} \leq \mu \leq 1; \\ 4\mu-3, if\ \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

Corollary 3.3: Putting $A=1, B=-1$ in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{2(2+\alpha)}{(3-\alpha^2)(2-\alpha)}-\dfrac{4}{(2-\alpha)^2}\mu, & if\ \mu \leq \dfrac{\alpha(2-\alpha)}{(3-\alpha^2)}; \quad (3.1) \\ \dfrac{2}{(3-\alpha^2)} & \text{if } \dfrac{\alpha(2-\alpha)}{(3-\alpha^2)} \leq \mu \leq \dfrac{2(2-\alpha)}{(3-\alpha^2)}; \quad (3.2) \\ \dfrac{4}{(2-\alpha)^2}\mu-\dfrac{2(2+\alpha)}{(3-\alpha^2)(2-\alpha)}, & \text{if } \mu \geq \dfrac{2(2-\alpha)}{(3-\alpha^2)}; \quad (3.3) \end{cases}$$

These estimates were derived by Choudhury C. [3] and are results for the class of univalent starlike functions.

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