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The Scientific, Mathematical And Astronomical Contributions Of Jamshīd Al-Kāshī: A Critical Analytical Study

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Abstract

This research is a critical and analytical study of the astronomical and mathematical works and contributions of Ghiyāth al-Dīn Jamshīd al-Kāshī, a great scientist of the Muslim world in medieval history. Al-Kāshī's works described here in this research include "al-Rīsalah al-Mūhītiyah" (The Treatise on the Circumference), in which he approximated π to 16 decimal places, and "Rīsalat al-Witr wal-Jaib" (Treatise on the Chord and the Sine) where he approximated sin 1 gradually up to 16 decimal places, both of which have been regarded as great feats. Among al-Kāshī's greatest works is "Miftah al-Hisab" (Key to Arithmetic). In one part of "Miftah al-Hisab", al-Kāshī extracted integer roots with a high degree of accuracy. Although al-Kāshī cannot be regarded as the first to use decimal fractions in his arithmetic computations, he brilliantly used a methodical way to deal with them, which is comparable with modern mathematical practices. Furthermore, al-Kāshī expanded $(a + b)^n$ in a way that was very similar to Newton's expansion, and he used a trigonometric table similar to Pascal triangle to get the expansion's coefficients.

Key words: "al-Rīsalah al-Mūhītiyah", "Rīsalat al-Witr wal-Jaib", "Miftah al-Hisab", al- Khaqānī Zīj, Nūzhet al-Hadāyik, Sūllam al-Sama

Introduction

Al-Kāshī was among the most prominent astronomers and mathematicians in the annals of the Persian history and additionally was one of the greatest scientists in the Muslim and the whole world medieval history. His works are not confined to astronomy and mathematics, but he also had a crucial role in advancing architecture, urban design and civil engineering. His contributions to science are too many to be covered in just one article of this size, and hence the objective of this research is to discuss how al-Kāshī and other Muslim scientists who preceded him filled a gap in astronomy and mathematics between the ancient scientists and the scientists of our time without which these two areas of science would have never progressed and reached the epic that we see now before our eyes.

According to Merzbach and Boyer (2011), mathematics and astronomy in the part of the world where Muslims controlled started to decline after the death of Nasir al-Dīn al-Tūsī, the great Muslim scientist of the 13th century, who was also Iranian like al-Kāshī, but the story of Muslim contributions to these two fields cannot be sufficient unless we cite the great works of al-Kāshī that enriched the scientific discoveries in the late 14th and the early 15th century. Later in his life, al-Kāshī moved to Samarkand to join a group of great Muslim scientists who were invited by Ulugh Beg, the ruler of Samarkand, to construct a great observatory there and establish a madrasa to make it one of the great centers of scientific learning. While he was staying in Samarkand al-Kāshī authored numerous books in both Persian and Arabic where he contributed a lot to various topics of astronomy and mathematics. The most impressive work that al-Kāshī produced in Samarkand was a textbook which he called “Miftah al-Hisab” (Key to Arithmetic) where he provided an elementary introduction to algebra and geometry and other branches of mathematics, and explained how they apply to astronomy, surveying, architecture, and trade and other fields of interest to be taught to students, scholars and professionals during that time.

Al-Kāshī was renowned for his genius computational skills which were perhaps matchless up to the era in which he lived. The accuracy of al-Kāshī’s computations seems to be unequaled with respect to the equations which he devised and which he managed to solve. He devised the approach now known as Horner’s method, where he employed decimal factions, which he perhaps borrowed directly from his Muslim predecessors and from the Chinese.[1]

[1] Merzbach, U. C. and Boyer, C. B. (2011). ‘A History of Mathematics’. 3rd Edition, Hoboken, New Jersey: *John Wiley and Sons Inc,* p.222.

The number of prominent Muslim mathematicians and astronomers before al-Kāshī was significantly larger than those who came after him and before the European Renaissance since those who came after him were much fewer. While Arabic learning started to wane, European scholarship was improving and ready to receive the intellectual legacy left by the astronomers and mathematicians who dominated during the mediaeval era.[2]

Methodology

The methodology of this research, which examines the scientific contributions of Jamshīd al-Kāshī, follows a critical, analytical and argumentative line of investigation. That is to say every single scientific contribution attributed to al-Kāshī and other Muslim scientists who preceded him will be rigorously examined regarding the role that they played in the creation of such a contribution, and such a line of investigation will be followed with respect to every contribution one by one till a fair and rational conclusion is reached, and the role that al-Kāshī played therein is determined and authenticated.

The rationale for this authentication (argumentative methodology of this research) is the assumption that learning and knowledge constitute an accumulative human experience and it is in most cases irrational to attribute the initiation of any part of knowledge solely to any single particular scholar, inventor or discover. Such a methodology is particularly adopted due to the fact that it has been observed that several writers, both Muslims and Westerners, sometimes tend to attribute the making of knowledge, or a considerable part thereof, to a single inventor or discoverer, and may be this is particularly true with respect to this research, the scientific contributions of a Muslim scientist.

However, we may notice from the outset, that adopting such a methodology is not easy and it may not at all feasible for one researcher to complete by himself alone in a single terse piece of work of the kind that is accomplished in this article. Nevertheless, if the initiation of this sublime objective is manageable, it is going to be both novel and outstanding.

[2] *Ibid*., p.222.

Contributions of Early Muslim Scientists

Talking about Muslim scientists' contributions to mathematics essentially means the period between 622 and 1600 AD, the Islamic golden era when Islam was both influential as a religion and a culture, which has also been the era in which Muslim scientists were active transferring scientific knowledge to other parts of the world form the Indians, Greek and Chinese and adding their own individual imprint.

Development of mathematics and astronomy, as tracked by historians from old times to the Renaissance, is sometimes defined as the Islamic period that took place from the eighth through the fourteenth centuries. At that time, most scientific activities were concentrated in Moorish Spain, the Middle East and the Arab world. While medieval Europe was languishing in the Dark Ages, the torch of scholarship was held by Muslim hands that kept it alight, and then passed it to Renaissance Europe.[3]

Mathematics which Carl Friedrich Gauss calls "the queen of sciences" played a significant role in the life of humans, and a world devoid of mathematics cannot be imagined. A great number of scholars from different regions of the globe throughout history contributed significantly to mathematics among whom were many Muslim mathematicians.[4]

To cover all the Muslim scholars' contributions to mathematics and astronomy is beyond this thesis, and instead the study will only focus on the accomplishments of Muslim scholars who preceded al-Kāshī, and specifically those who helped him acquire scientific knowledge and use it to promote his own and add to it. The researcher examined such historical scholarships to truly appreciate how excellent they are in relation to the period of their accomplishment, as some topics that are common knowledge today, were probably not known by the scientist of that time whom we describe as genius.[5]

Thus the period in which transformation of ancient books took place, such as Greek, Chinese, and Indian texts into Arabic extend from the seventh to the seventeenth century, and

[3] Gingerich, O. (1986). 'Islamic Astronomy'. *Scientific American, Inc,* p.74.
[4] Kadyrov, S. (2009). 'Muslim Contributions to Mathematics'. *The Fountain*, (67), January and February, p.1.
[5] *Ibid.,* p.1.

hence comprise major historical periods that might be utilized to demonstrate how Islamic scientific studies have developed and declined. However, the time in which the development of creative Islamic thought took place between the ninth and eleventh centuries, also constitutes the period in which translation, transfer and inscription of Greek, Chinese, and Indian texts into Arabic were in full swing. In the latter period Muslims were responsible for preserving, advancing, and ultimately transmitting ancient knowledge to Europe. Finally, with a brief explanation of the findings of Muslims in astronomy and mathematics, this research is going to make an attempt to realize the objective of describing the contributions of a Muslim scholar in science to the entire world.[6]

Not long after the birth of Islam, Muslim scholars started to acquire Greek expositions and set up their study and translation into Arabic. They thoroughly analysed, organized, approved, and advanced Greek science and philosophy. The age called Golden Age of Islam that flourished for fewer than three centuries, started essentially after this time. This was the time when numerous Muslim scientists produced many books on various areas of science.[7]

Al-Kāshī's Early Life.

Ghiyāth al-Dīn Jamshīd Masūd al-Kāshī, also known as Jamshīd Kāshāni, was one of the most renowned mathematicians and astronomers in Persian history as well as one of the most fascinating mediaeval mathematicians in the world. He was born, nurtured, and educated in the little town of Kāshān, which is located in central Persia, in the Tamerlane Empire, at some time in the last quarter of the fourteenth century. Although some records indicate he was born about 1380, his precise birth date is uncertain. In the early hours of the morning of Wednesday, 22 June, 1429 (19 Ramadan, 832 AH),

[6] Afridi, M. A. (2013). 'Contribution of Muslim Scientists to the World: An Overview of Some Selected Fields'. *Revelation and Science,* 03(01), 47-56.

[7] Armstrong, K. (1991). 'Holy War: The Crusades and their Impact on Today's World'. *New York: Doubleday*, 64-65, 225-226.

al-Kāshī died at the observatory in the outskirts of Samarkand, in what is now modern-day Uzbekistan, in Central Asia.[8]

It is widely believed, based on a number of sources and earlier research, that al-Kāshī spent the greatest part of his life pursuing his interests in astronomy and mathematics and creating new things. Born into a low-income family, he travelled much in his early years in pursuit of making a living but never stopped to pursue his passion for learning. For this reason, he sought funding from sponsors who found value in his work and dedicated his first work on astronomy to Kamaleddin, a statesman who served as a minister to King Shah Rukh, the son of Tamerlane who took over the kingdom and established Samarkand as his capital. He dedicated his second treatise to Ulugh Beg, the grandson of Tamerlane who inherited Shah Rukh, his father, and ascended the throne. He then dedicated his third treatise to Sultan Iskender, a vizier under Ulugh Beg. Although, it appears that al-Kāshī sought favour by dedicating his treatises to the members of the royal family, he was extremely careful not to indicate any preferences.[9]

During his early youth, al-Kāshī was thirst for knowledge and sought to befriend many famous scientists from whom to take scientific knowledge. Among these was Qade Zade al-Rūmi who was very generous to impart on his students all the knowledge that he had. The two used to sit together frequently for hours to discuss and solve problems of Mathematics and astronomy.[10] Furthermore, Ulugh Beg became by this time a passionate patron of architecture beside mathematics and astronomy. This was also the time (1417) when Ulugh Beg established his famous madrasa in Bukhara, another major city of his kingdom, where science was also taught.[11]

[8] Azarian, M. K. (2019). 'An Overview of Mathematical Contributions of Al-Kāshī (Kāshānī)'. *Mathematics Interdisciplinary Research*, 4, 11-19.
[9] Kennedy, E. S. (1960b). 'The Planetary Equatorium of Jamshīd Ghiyāth al-Dīn al-Kāshī'. Princeton, New Jersey: *Princeton University Press*, 2.Bartold, V. V. (1963). 'Four Studies on the History of Central Asia'. Vol. II, translated by V. Minorsky, Leiden, Netherlands: *Brill Academic Publishers*, p.130.
[10] Sayili, A. (1960a). 'Giyāth al-Dīn al-Kāshī's Letter on Ulugh Beg and the Scientific Activity in Samarkand'. *Ankara: Turkish History Society*, pp.13-15.
[11] Knobloch, E. (1972). 'Beyond the Oxus'. London: *Ernest Benn Ltd*, p.164.

Simultaneously the madrasa of Samarkand became more developed when an observatory was built nearby in 1424.[12] Ulugh Beg also realized at this time that astronomical tables were necessary to support astronomical works in the observatory. As soon as Ulugh Beg became convinced of al-Kāshī brilliance, he invited him to help with the observatory's construction. Qade Zade did not fail to advise Ulugh Beg to call al-Kāshī to work in the observatory and the school.[13]

At some time during the period 1417-1420 Ulugh Beg established a madrasa (school) dedicated to advanced studies in religion and science - which still exists today as a visible witness of the great scientific accomplishments that have once been flourishing in Samarkand as well as among the great spectacular archeological and historical buildings in Uzbekistan. According to Kary-Niyazov (1967), the madrasa (school) represented a prelude to the construction of the observatory which started in 1424, but unfortunately both the observatory and the madrasa were completely neglected after Ulugh Beg expired in 1449 and were only rediscovered in 1908. With these great feats in his mind, Ulugh Beg called many scientists (60 according to an account by al-Kāshī in a letter he wrote to his father) to teach science in the madrasa (school) and then in the construction of the observatory including al-Kāshī himself. In about 25 years from the time when the observatory construction works started in in 1424 till the unfortunate assassination of the ruler Ulugh Beg in 1449, Samarkand was at the center of the annals of astronomical and mathematical activities led by none but al-Kāshī the greatest mathematician and astronomer of that time aided by two prominent astronomers, namely Qaḍe Zade al-Rūmi, and another scientist from Kāshān called Muin al-Dīn, and all under the sponsorship and patronage of Ulugh Beg himself.[14]

Once in Samarkand, al-Kāshī actively resumed his mathematical and astronomical works while taking a leading role in building the observatory, where he was in charge of

[12] Bartold, V. V. (1963). 'Four Studies on the History of Central Asia'. Vol. II, translated by V. Minorsky, Leiden, Netherlands: *Brill Academic Publishers*, p.119.

[13] Bartold, V. V. (1963). 'Four Studies on the History of Central Asia'. Vol. II, translated by V. Minorsky, Leiden, Netherlands: *Brill Academic Publishers*, p.130.

[14] Kary-Niyazov, T. N. (1967). 'Ulugh Beg's Astronomical School' 2nd ed., Tashkent. *MS Publishing*, pp.148-325.

providing the most advanced equipment, and at the same time writing his own Zīj which he named “al-Khaqānī Zīj” (Al-Khagani Astronomical Tables), and these same tables were then developed to create “Ulugh Beg’s Zīj”, in which other astronomers participated and completed after al-Kāshī’s death. As such al-Kāshī used to play a significant role relative to other members of the scientific staff of Ulugh Beg. In fact, al-Kāshī was second to none in the activities of the observatory and the madrasa except Ulugh Beg the governor of Samarkand and the patron of the entire scientific staff whom he invited to construct the observatory and teach in the madrasa; and for this reason al-Kāshī was dubbed “the second Ptolemy” and the “Support of Astronomical Science” and “Maulana-i-Alam”.[15]

There is some evidence to show that al-Kāshī took a respite after 1416 while in Samarkand and disappeared from the public eye for about 5 years, because he did not produce any significant work during these years, although Samarkand was by this time the hub of great scientific activities. Meanwhile, a multitude of scientists came to work for Ulugh Beg in Samarkand from every corner of the globe. Aside from being a scientist, Ulugh Beg was also a politically self-dependent ruler in his historical region of Central Asia[16].

By 1420-1421, al-Kāshī established himself in Samarkand as both a mathematical teacher and an astronomer.[17] In a letter he wrote to his father, al-Kāshī informed him that he had been offered the position of overseeing the work of the other scientists in building the observatory. Ulugh Beg then ordered the building of Suds-i-Fahri, a massive meridian arch or "geometrical platform" based on al-Kāshī’s recommendations. The construction of the observatory and the platform was wrapped up in only three years, and by 1423 the giant structure was ready for astronomical work[18].

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[15] Bartold, V. V. (1963). ‘Four Studies on the History of Central Asia’. Vol. II, translated by V. Minorsky, Leiden, Netherlands: *Brill Academic Publishers*, p.88.

[16] Ozdural . A. (1990). Giyaseddīn Jemshid el-Kāshī and Stalactites, Middle East Technical University Journal of the Faculty of Architecture 10 (1990), pp.31-49.

[17] Sayili, A. (1960b). ‘The Observatory in Islam and its Place in the General History of the Observatory’. *New York: Arno Press,* 11-12.

[18] *Ibid*., p.11.

Shortly after this, in July 1424, al-Kāshī finished " al-Rīsalah al-Mūhītiyah" (Treatise on the Circumference), in which he amazingly accurately calculated π value. That the dates of these brilliant works came on the heels of each other may not be by accident; perhaps because much of the calculations of the meridian arch are related to the relation between the circumference and circle's diameter. In the letter al-Kāshī wrote to his father, he tells us that the making of the meridian arch was a great challenge to him, and hence he calculated the arch with a high precision, which also aided in writing the circumference treatise.

All through his active life, al-Kāshī managed to reconcile between his profession as astronomer and his writing and teaching activities. Thus, in June of 1426 he completed his second book which he called "Nūzhet al-Hadāyik" (Delight of the Gardens). On the 2 of March, 1427 he finished his major work which he called "Miftah al-Hisab" (Key to Arithmetic), which he authored as a textbook that included an elementary introduction to arithmetic, trigonometry and algebra, to be used by astronomers, mathematicians, engineers, architects, surveyors and students and presented it to Ulugh Beg, his patron.[19] This effort of al-Kāshī had an exceptional value in the annals of science and perhaps may be a great deed in its own right as it contained solutions for arithmetical problems, the binomial theorem, methods of roots extraction, and mathematical series, and many other basics of mathematics. In this textbook he used both the sexagesimal and decimal systems, while the decimal system was not yet introduced to the scientists of Europe till 1585. He also presented problems in plane and solid geometry and described arches, cones and domes[20].

In his final work, "Rīsalat al-Witr wal-Jaib" (Treatise on the Chord and the Sine", al-Kāshī outlined and illustrated an iterative approach by which to calculate the sin of 1^{o} degree to a high degree of precision. But unfortunately he passed away on the 22 June, 1429 before finishing this work while still

[19] *Ibid*., p.6.

[20] Vernet, J. (1974). 'Al-Kāshī, Encyclopedia of Islam'. (IV), Leiden, Netherlands: *Brill Academic Publishers*, 703-4. Schirmer, H. (1936). 'Misaha, Encyclopedia of Islam'. (III), Leiden, Netherlands: *Brill Academic Publishers*, pp.517-19.

working to add the final touches of the observatory just outside outskirts of Samarkand[21].

In addition to these major works, al-Kāshī authored six treatises which were shorter and about less important topics. Among these treatises there was one which illustrates how to determine the "Kibla", i.e. the orientation a Muslim takes when he performs prayers, and hence is of special importance to practicing Muslims. One of the descendants of al-Kāshī by the name of Al-Rezzak completed a manuscript of Miftah al-Hisab on 14 August, 1589.

Al-Kāshī's Personality

Al-Kāshī was lauded by his counterparts and dubbed 'the second Ptolemy' and the generation that came after them called one mathematician of their time "the second Giyāth al-Dīn Jemshīd". [22] He was certainly a great scientist and a mathematician with exceptional capabilities, as well as an observer with a keen eye for detail, a resourceful inventor, a brilliant innovator, and a productive author. But a lot of evidence is needed in order to have a meticulous evaluation of his works and his genius, which we are trying to do here.

The letter that he sent to his father had a few clues that describe his personality and character. The letter is full of self-praise and unhidden predominance over his counterparts and colleagues. However, if we just judge by the great works he did, we will see how right he was, not only that but what he wrote in his lifetime was superior to many of his contemporaries and his predecessors. Most of his works was unique and we can easily judge that he was ahead of his time, as it took lots of time for other scientists to better his accomplishments.

But those who used the letter sent to his father to prove his vanity and protestations were sometimes exaggerating, and they often used the few hints contained in the letter to prove his vanity. However, the only true fact that can be inferred here is that al-Kāshī was very proud of his accomplishments and very keen to demonstrate his prevalence. He was so obsessed with self-esteem that at times he indeed adored himself. A scientist

[21] Kennedy, E. S. (1960b). 'The Planetary Equatorium of Jamshīd Ghiyāth al-Dīn al-Kāshī'. Princeton, New Jersey: *Princeton University Press*, pp.6-7.
[22] *Ibid*., p.9.

of his weight must have definitely been after distinction, and to this effect he was very eager to boast that much of the scientific knowledge of his time were accumulated together in Samarkand, and he often embodies his personal prevalence by minimizing his colleagues[23]. If he did some of his coworkers at the observatory their dues, it was Qade Zade, but this one was his tutor for many years, and al-Kāshī cannot deny the knowledge that he extracted from him[24]. But al-Kāshī still described Qade Zade on many occasions as a scientist who was just beginning to understand theoretical astronomy[25].

In the letter that he sent to his father, al-Kāshī admitted that he gave Ulugh Beg false information regarding the Meragha Observatory that has been constructed by Nasir al-Dīn al-Tūsī. In order to persuade Ulugh Beg of the benefit of large astronomical instruments, he gave high value to the Suds-i-Fahri meridian arch that he believed to be great and magnificent. He falsely claimed that a geometrical minber called Suds-i-Fahri had been built in Meragha Observatory in order to persuade Ulugh Beg to allow its construction in Samarkand[26]. Such a device wasn't there in the observatory made by Nasir al-Dīn al-Tūsī in Meragha. Wilber[27](1969, p. 10) reported that the meridian's height was recorded through sun rays entering by a small opening in the building's dome that then falls on the pavement. Ulugh Beg placed a high value on the Suds-i-Fahri meridian arch, which he described as being enormous and splendid, in order to persuade Ulugh Beg of the benefit of large astronomical instruments.

The kind of al-Kāshī's character which we believe is well-established and proven by many sources, make us convinced that al-Kāshī was resourceful and quick-witted, two attributes that he used to employ in order to assert his status as one of the great court members of Ulugh Beg. But it cannot be denied that he often asserted himself as a man of bad manners who may not hesitate before he trashes his colleagues in front of a big audience; a person of many talents who later in his career

[23] Sayili, A. (1960b). 'The Observatory in Islam and its Place in the General History of the Observatory'. *New York: Arno Press*, pp.44-48.
[24] *Ibid*., p.107.
[25] *Ibid*., p.107.
[26] *Ibid*., p.98.
[27] Wilber, D. N. (1969). 'The Architecture of Islamic Iran'. *New York: Greenwood Press,* p.10.

became active in architecture only to advance his personal interests.[28]

Al-Kāshī's Works.

Al-Kāshī was renowned for his amazing aptitude at solving puzzles that required complex mental calculations. For this reason he was given many nicknames pointing to this aptitude including "The Reckoner", "The Second Ptolemy", "The King of Engineers", "The Pearl of the Grandeur of His Age", and "Our Master of the Globe", among others. Originally al-Kāshī was a physician, especially during the prime of his youth, where he used to treat minor illnesses based on traditional medical practices. However, he soon became recognized as a mathematician and astronomer with great passion in both fields. Like many scientists of his time, he wrote most of the known scientific treatises as a means of sustenance and dedicated them to the rulers of his time or other statesmen, in order to maintain himself financially[29].

Despite being Persian, al-Kāshī wrote the majority of his mathematical works in Arabic in order to broaden the base of his readership. As a result, al-Kāshī originally wrote all of his works by hand, mostly in Arabic and sometimes in Persian, and mostly by himself. However, the majority of his great works have now been rendered into Russian, French, English, German, and many other languages by translation or summation.[30]

Al-Rīsalah Al-Mūhītiyah.

Al-Kāshī personally wrote "Al-Rīsalah Al-Mūhītiyah" (The Treatise on the Circumference) in 58 pages in Arabic, and completed it in July 1424 (Sha'ban 827 AH) in Samarkand. The manuscript of Al-Rīsalah Al-Mūhītiyah was presented by King Nader Shah to the Qudse Razawi library in Mashhad, Iran, in

[28] Sayili, A. (1960a). 'Giyāth al-Dīn al-Kāshī's Letter on Ulugh Beg and the Scientific Activity in Samarkand'. *Ankara: Turkish History Society*, pp.101-102.
[29] Azarian, M. K. (2019). 'An Overview of Mathematical Contributions of Al-Kāshī (Kāshānī)'. *Mathematics Interdisciplinary Research*, 4, 11-19.
[30] *Ibid*., pp.11-19

1734 (1145 AH). The manuscript contains an introduction, 10 chapters and a conclusion.[31]

The value of π was calculated by several mathematicians includingArchimedes (c. 287-212 BC), Abūl Wafa al-Būzjānī (940-998 AD), and Abu Rayhān al-Birūnī (973-1048 AD). These approximations motivated al-Kāshī to write a treatise with the aim of improving the value of π that was estimated by earlier great mathematicians. The need for more accurate trigonometric tables along with cutting-edge astronomy research served as another inspiration for him to produce this dissertation. What we shall refer to as al-Kāshī fundamental theorem was the primary instrument for calculating and is as follows:

As adapted from the calculations of Azarian (2004), we may illustrate al-Kāshī's fundamental theorem as follows:

If we assume an arbitrary chord AG on a semicircle with diameter AB = 2r and center E, and we designate the point D as the center of the chord GB (figure 1), then r (2r + AG) = AD.[2]

Four separate premises from Euclid's Elements serve as the foundation of al-Kāshī's demonstration of this fundamental theorem [32] . Nevertheless, al-Kāshī spared the reader the burden of discussing Euclid’s method of the calculation of π which he did not mention in his theory. This is because Euclid's Elements at that time were well-known and frequently utilized in geometrical arguments. We employed modern notations and the right angled triangle of the Pythagorean Theorem as an alternative demonstration of al-Kāshī's fundamental theorem.[33]

After more discussion, he precedes that if the chord AG in a unit circle (figure 1) is equal to 2 radians, then by using the right angled triangle AGB we can see that AG = 2 sin 1. The right angled triangle ADB can be obtained in a similar way:

[31] Azarian, M. K. (2019). ‘An Overview of Mathematical Contributions of Al-Kāshī (Kāshānī)’. Mathematics Interdisciplinary Research, 4, 11-19.
[32] Azarian, M. K. (2004). ‘Al-Kāshī’s Fundamental Theorem’. *International Journal of Pure and Applied Mathematics*, 14(4), pp.499-509.
[33]*Ibid*., pp.499-509

$$AD = 2\sin\frac{1}{2}\left(\theta + \frac{\pi - 2\theta}{2}\right) = 2\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Then he substituted AG, AD and r = 1 in his fundamental theorem to obtain:

$$2 + 2\sin\theta = 4\sin^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

]That is to say:

$$\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \sqrt{\frac{1 + \sin\theta}{2}}$$

Where $\theta < \frac{\pi}{2}$

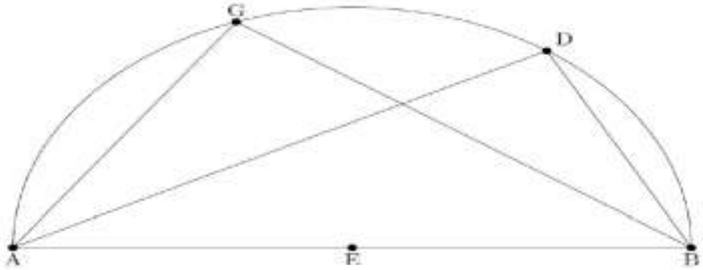

Figure 1

This will be what is now called Lambert identity, which uses the trigonometric method of al-Kāshī's fundamental theorem. It should be noted here that al-Kāshī established this identity before 1424 (827 AH), or more than 300 years earlier than the period in which Johann Heinric Lambert (1727-1777) lived. Thus, it only appeared in Lambert's book in 1770 i.e. about 346 years later. To develop a simple identity of this kind with such impressive applications requires a genius.

By discovering such a high accuracy of π in the middle ages, al-Kāshī's fundamental theorem offered him a great lead over both his contemporaries and predecessors. He used his theorem to compute the spans of regular polygons that are inscribed in a circle with each having 3×2^n sides ($n \geq 1$).

Next, al-Kāshī managed to calculate the number of the sides of a regular polygon inscribed in a circle, such that its radius is 600,000 times the radius of the Earth, and with such high accuracy that the difference between the circle’s circumference and the perimeter of a regular polygon inscribed in this circle is as small as a horse's hair width or even less. Al-Kāshī used his fundamental theorem to calculate π accurately to 16 decimal places employing the case of an inscribed polygons of 3×2^{28} sides or 805,306,368 sides.

Prior to al-Kāshī, Tsu Chung-Chih, the Chinese scientist approximated π, about the year 480, accurately to six decimal digits, which represent the best approximation till that time. As of now we know almost 22 trillion (actually 22,459,157,718,361) decimal digits of π as of 11 November, 2016; a feat recorded by Peter Trueb (2016), which makes the estimation of π by al-Kāshī seems common place. Nevertheless, al-Kāshī's estimation of π to 16 decimal places in 1424 was definitely a remarkable feat, because in doing so he outperformed all other mathematicians till that time, starting from Archimedes and Ptolemy[34].

In 250 BC, Archimedes' approximation of π produced 3.14, while in 150 AD Ptolemy's calculation of π produced 3.14166. 155 years passed after al-Kāshī's determination of π, when work on the approximation of the value of π was resumed again by François Viete in 1579, but to obtain a meager result of accurate 9 decimal points only. Adrian Romain obtained an approximation to 15 decimal places in 1593; still less than that of al-Kāshī; and only Ludolf van Ceulen managed to improve the result to 35 decimal places in 1610, i.e. 172 years passed before European scientists were able to surpass al-Kāshī's estimation of π. It's important to note that in 1665, Isaac Newton calculated π with a precision of only up to 16 decimal digits using arctan x power series expansion[35].

Rīsalat Al-Witr wal-Jaib

Of the three of al-Kāshī's most important mathematical works, "Rīsalat al-Witr wal-Jaib" (Treatise on the Chord and the Sine) may be the most brilliant one because of its vast applications. The main topic of discussion in this treatise is how to estimate the sine of one third of an angle whose sine and chord are known. This work is thought to have been completed by al-Kāshī between the years 1424 and 1427. Unfortunately, the manuscript has been lost. However, because the main objective of this treatise was to calculate sin 1, some of al-Kāshī's contemporary scientists and successors wrote his works in both Persian and Arabic about the sin 1 calculation driven by al-Kāshī's iteration approach. We will base our

[34] Azarian, M. K. (2019). 'An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)'. *Mathematics Interdisciplinary Research*, 4, 11-19.
[35] Bailey, D. H., Borwein, J. M., Borwein, P. B. and Plouffe, S. (1997). 'The Quest for Pi'. *Math Intelligencer*, 19(1), 50-57.

discussion in this matter on “Sharh-i Zīj-i Ulugh Beg” (Comments on Ulugh Beg's Astronomical Tables) written by al-Birjāndi (Nizam al-Dīn al-Birjāndi; died 1528 AD), the Persian mathematician and astronomer[36].

According to Azarian[37] (2004), al-Kāshī calculated sin 1 in two steps; first he created his renowned cubic equation by applying Ptolemy's theorem to a quadrilateral inscribed in a circle. He then devised a clever and speedy iteration process to calculate sin 1 with an accuracy of 16 decimal digits (nine accurate sexagesimal places) employing the cubic equation’s root. Both geometry and algebra were used by Al-Kāshī to approximate sin 1 to a high accuracy relative to his predecessors, which was considered a marvelous accomplishment at that time. This wasn't only the most intriguing and original approximation technique, but was indeed a unique approximation one in the history of trigonometry and the pinnacle of medieval algebra. Additionally, this was the highest approximation of sin 1 during that era.

Two earlier Muslim scientists, Abūl Wafa al-Būzjānī (940-998), and Abūl Hasan Ibn Yūnūs (950-1009), obtained in the tenth century the highest approximations of sin 1 that were accurate to 4 sexagesimal places. Al-Kāshī's determination of sin 1 in this way wasn't outdone until Taqi al-Dīn Mohammed al-Asadi (1526-1585) offered a better result in the 16th century.

Sin 1 was calculated by al-Kāshī using the sexagesimal system. However, in this research sin 1 is calculated using the decimal format because we are now more accustomed to it.

The following calculation of sin 1 by al-Kāshī is adapted from Azarian (2004), and we may illustrate it as follows:

Let A, B, C, D be points plotted on a semicircle with radius r and center F (figure 2) such that:

$$AB = BC = CD$$

[36] Azarian, M. K. (2019). ‘An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)’. *Mathematics Interdisciplinary Research*, 4, 11-19.
[37] Azarian, M. K. (2004). ‘Al-Kāshī’s Fundamental Theorem’. *International Journal of Pure and Applied Mathematics*, 14(4), 499-509.

Then by Ptolemy's theorem we will have:

$$\mathrm{AB.CD} + \mathrm{BC.AD} = \mathrm{AC.BD}$$

Since:

$$\mathrm{AB} = \mathrm{CD} = \mathrm{BC} \text{ and } \mathrm{BD} = \mathrm{AC}$$

We obtain equation (1):

$$\mathrm{AB}^2 + \mathrm{BC.AD} = \mathrm{AC}^2$$

Then we determine a point G that lies on the diameter AE such that EC = EG. We can get from the two similar isosceles triangles ABG and ABF that:

$\frac{\mathrm{AB}}{\mathrm{AG}} = \frac{\mathrm{AF}}{\mathrm{AB}}$ and $\mathrm{AG} = \frac{\mathrm{AB}^2}{\mathrm{r}}$ and thus $\mathrm{EG} = 2\mathrm{r} - \mathrm{AG} = 2\mathrm{r} - \frac{\mathrm{AB}^2}{\mathrm{r}}$

From the right angled triangle we get equation (2):

$$\mathrm{AC}^2 = \mathrm{AE}^2 - \mathrm{EC}^2 = 4\mathrm{r}^2 - \mathrm{EG}^2$$

From (2) we get equation (3):

$$\mathrm{AC}^2 = 4\mathrm{r}^2 - \left(2\mathrm{r} - \frac{\mathrm{AB}^2}{\mathrm{r}}\right)^2 = 4\mathrm{AB}^2 - \frac{\mathrm{AB}^4}{\mathrm{r}^2}$$

From equation (1) and equation (3) we get:

$$\mathrm{AB}^2 + \mathrm{AB.AD} = 4\mathrm{AB}^2 - \frac{\mathrm{AB}^4}{\mathrm{r}^2}$$

And from the above equation we get equation (4):

$$\mathrm{AD} = 3\mathrm{AB} - \frac{\mathrm{AB}^3}{\mathrm{r}^2}$$

If AB = crd 2α, then AD = crd 6α; and from (4), we get equation (5):

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

To get al-Kāshī's famous cubic equation (6) we let α = 1° and x=sin 1° and hence:

$$\mathrm{x} = \frac{4}{3}\mathrm{x}^3 + \frac{1}{3}\sin 3^{\mathrm{o}}$$

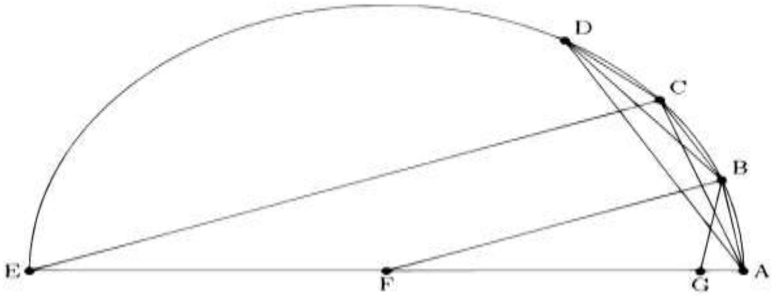

Figure 2

what he added or improved in those works that belong to Muslim mathematicians who preceded him. Miftah al-Hisab was primarily written as a textbook and at the same time an encyclopedia, and as such was used by many generations of mathematicians, astronomers, students, accountants, merchants, engineers, architects, land surveyors, and other professionals, and even astrologers for as many as five centuries, till new text books were created. Al-Kāshī completed the book on 2 March, 1427, (3 Jumada I, 830 AH), taking from him more than 7 years to complete. He created this exceptional mathematical work to support himself financially and hence and dedicated it as expected to Ulugh Beg, his patron and ruler of Samarkand[38].

Miftah al-Hisab is made up of five books with different topics and is preceded by an introduction. Each book has a number of chapters, which are entitled as follows: "On the Arithmetic of Integers", (six chapters), "On the Arithmetic of Fractions", (twelve chapters), "On the Computations of Astronomers", (six chapters), "On the Measurement of Plane Figures and Bodies", (nine chapters) and "On the Solutions of Problems by Means of Algebra", (four chapters).

Al-Kāshī opens Miftah al-Hisab with a book entitled "On the Arithmetic of Integers" in which he explains a general detailed technique for obtaining any integer's root in the decimal and sexagesimal systems. The technique of this book was only discovered by European mathematicians in the 19th century and its approach reformulated and named after Ruffini-Horner and became known as Ruffini-Horner’s approach. Al-Kāshī made advantage, in this book, of the works of numerous Persian mathematicians including Omar al-Khayyām, Abu Rayhān al-Birūnī, and Nasir al-Dīn al-Tūsī who attempted to extract integer roots with some degree of success. However, al-Kāshī asserted in this book his pioneering discovery of this technique including that of the calculation of the integer roots[39].

[38] Azarian, M. K. (2019). ‘An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)’. *Mathematics Interdisciplinary Research*, 4, 11-19.

[39] Azarian, M. K. (2019). ‘An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)’. *Mathematics Interdisciplinary Research*, 4, 11-19.

Both the second and third books are focused on decimal fractions which al-Kāshī previously used in the "Treatise on the Circumference". Although some sources credited al-Kāshī with the introduction of decimal fractions in his arithmetic computations, he is no way the first to introduce them even among the Arab mathematicians. They have been used earlier by Chinese and Indian mathematicians as indicated by al-Ūlidīsī in his "Treatise of Arithmetic", noting that al-Ūlidīsī lived in the mid-tenth century[40]. The actual credit that may go to al-Kāshī is that he brilliantly used the decimal fractions in a methodical way, by which he meant to introduce a method where decimal fractions would replace the sexagesimal fractions in all calculations and thereby he created an arithmetic system that can replace the sexagesimal system for all integers. This system is primarily based on the decimal numeration, and in this manner is much easy for those not used to the somewhat difficult sexagesimal system used by astronomers.

Al-Kāshī studied in this book the decimal fractions and explained them fully, but his eyes did not fall on the cases of periodicity, which perhaps he did not notice. In cases in which the number is made up of an integer and a decimal fraction, he separated them by a vertical line. The late 15th century and the early 16th century saw some circulation of al-Kāshī's approach of decimal fractions in Turkey, which may have been conveyed to Turkey by Ali Qūshji who was a Turkish scientist invited by Ulugh Beg to work in the observatory as well as a colleague of al-Kāshī while working together on the observatory of Samarkand. Ali Qūshji is known to have settled in Istanbul after the brutal death of Ulugh Beg and the conquest of Constantine by the Ottomans. Al-Kāshī's approach to decimals also appears in a collection of some Byzantine problems by an unknown writer who wrote them some time in the late 15th century, and reached Vienna in the year 1562. This may also indicate that al-Kāshī's system of decimal fractions may have also reached other countries of Europe later on[41].

The fifth book of Miftah al-Hisab deals with fourth-degree equations which he dealt with under a special title written as "The Method of Computation of Unknowns" where he made attempts to solve seventy problems which he claims have

[40]Saidan, A. (1966). 'The Earliest Extant Arabic Arithmetic'. *Oxford: Isis Publishing Ltd*., 57, 475-490.
[41]Hunger, H. and Vogel, K. (1963). 'A Byzantine Arithmetic Book of the 15th Century'. *Sciences in Vienna*, p.78.

never been known by any of his contemporaries or predecessors. He expressed his further intention to elaborate on this part of the book, but it seems that he changed his mind and satisfied himself with what has been already done. It also seems that the manner in which al-Kāshī dealt with fourth-degree equations have been dealt with by al-Khayyām and those mathematicians who came after him in the 11th century, where fourth-degree equations and their positive roots were constructed as coordinates of intersections of conic shapes pair points. Although al-Kāshī expressed his intention to deal with 70 problems related to fourth-degree equations not touched by others, he actually made a few attempts on only sixty-five. However, there is ample proof to show that al-Kāshī was the first to deal extensively on the fourth-degree equations relative to those who preceded him.[42]

Extraction of Roots

Al-Kāshī explains the process of root extraction in Miftah al-Hisab. He brings to our attention the cycles concept that he will use later in the iterative process of root extraction. He then describes the algorithm of extraction step by step and offers the layout in general for any arbitrary number. He executes the algorithm using a table, but provides a description which is purely in prose. He illustrates the algorithm by giving a few examples, but without indicating why he follows the steps that he takes every time[43].

The iterative approach that al-Kāshī used is very much like the approach we have been using in the elementary school to teach pupils how to extract roots. For example, if we want to obtain the square root of 33, we first look for an integer whose square is nearest to but less than 33. This will be 5 which when we square it, it will give us 25, and then the remainder will be 8. We now proceed to get the fraction that when added to 5 will give a more accurate square root of 33. We enter two

[42]Taani, O. (2014). 'Multiple Paths to Mathematics Practice in Al-Kāshī's Key to Arithmetic'. *Science & Education*, 23(1), 125-141.
[43]Dakhel, A. (1960). 'The Extraction of the n-th Root in the Sexagesimal Notation: A Study of Treatise 3 of Miftah al-Hisab', Edited by W. A. Hijab, and E. S. Kennedy, Beirut: *American University of Beirut.* Rashed, R. (1994). 'The Development of Arabic Mathematics between Arithmetic and Algebra', (Trans: Armstrong, A.F.W.), Dordrecht: *Kluwer,*p.35.

decimal places to get 8.00 and we double the 5 to get 10 and again look for an integer whose square is nearest to but less than 8.00, which will now be 7 which when multiplied by 107 will give 749, and in this way the first decimal place will be 7 and the result will be 5.7. We subtract 800 minus 749 and get 51 and we enter another two decimal places in order to have 5100. We now double the 57 to get 114 and look for an integer which when multiplied by 114 will be nearest but less than 5100, which in this case will be 4, i.e. the whole number will be 1144 and its multiplication will give 4576 and the second decimal place in this case will be 4 and in this way the result will be 5.74. We can repeat this iterative process infinitely, but we should note that the operation will be every time more and more tedious as the numbers get bigger and bigger.

Al-Khaqānī Zīj

In Persian, Zīj is a term used for books that contain astronomical tables which are used in other astronomical computations. Al-Kāshī finished al-Khaqānī Zīj (Al-Khaqani Astronomical Tables) around 1413-1414 before moving to Samarkand. The tables, which he wrote in Persian, use "il-Khānī Zīj" that was compiled some 150 years earlier by Nasir al-Dīn al-Tūsī, the great Muslim Persian astronomer of the 13th century. Bartold (1963) indicates that al-Kāshī dedicated the tables to Shah Rukh, the first ruler of the empire and Ulugh Beg's father, who was at that time in Herat, another major city in the empire. However, other sources including Kennedy (1960a) assert that al-Kāshī dedicated the tables to Ulugh Beg, which is more plausible. Since al-Kāshī wrote this work before he went to Samarkand, he must have sought Ulugh Beg's patronage several years before moving to Samarkand and from a far distance so as to secure funding. The book of the tables includes six treatises all computed in the sexagesimal system. Al-Kāshī, not only admits that he used the il-Khānī Zīj of al-Tūsī, but he also admits that he used the tables of earlier astronomers along with their geometric proofs.[44]

Sūllam Al-Sama

Sūllam al-Sama, variously translated as "The Ladder of the Sky" or "The Stairway of Heaven", is the earliest work known to

[44]Youschkevitch, A. P. and Rosenfeld, B. A. (1973). 'Al-Kāshī in Dictionary of Scientific Biography'. *New York: Charles Scribner's Sons,* 7, 255-262.

have been done by al-Kāshī. Al-Kāshī finished this work in 1407 in Kāshān, his hometown. This book was originally named "Solution for Difficulties in Determining Sizes and Distance of Heavenly Bodies". Al-Kāshī included the whole treatise of Sūllam al-Sama in Miftah al-Hisab's preface. As the original title indicates, al-Kāshī corrected in this book the mistakes committed by earlier astronomers regarding sizes of planets and stars and distances between them, and also settled the disagreements between those earlier astronomers. The essence of this book was to help students and future astronomers solve astronomical problems[45].

Nūzhet Al-Hadāyik

Nūzhet al-Hadāyik (Delight of the Gardens) is a treatise that al-Kāshī started to write on 10 February, 1416, in Kāshān, his hometown, but he edited the treatise and added some sections to it while in Samarkand and finally produced it in 1426. There exists a manuscript of this treatise written in Istanbul around 1490 in Persian language by an unknown scientist. The main part of the treatise is about an instrument called "The Plate of Conjunctions" that al-Kāshī built while in Kāshān, and the treatise explains how the instrument works. The instrument is described by Al-Kāshī as one that can be used by astronomers to get values of celestial distances and sizes and distances between stars and the earth, in addition to the dates of eclipses and motion of stars and planets. The instrument looks very much like the astrolabe in terms of shape and functions, and therefore can describe the movement of stars and planets based on their mean coordinates. Using this instrument, al-Kāshī managed to estimate accurately the eclipses that occurred in the three years of 809-811 AH (1407-1409 AD)[46].

[45] O'Conner, J. J. and Robertson, E. F. (2020). 'McTutor History of Mathematics Archive'. *Advances in Pure Mathematics*, 10(8), August 11, 2020.
[46] Kennedy, E. S. (1960a). 'A Letter of Jamshīd al-Kāshī to His Father: Scientic Research and Personalities at a Fifteenth Century Court'. *Orientalia,* 29, 191-213. Vernet, J. (1974). 'Al-Kāshī, Encyclopedia of Islam'. (IV), Leiden, Netherlands: *Brill Academic Publishers*, 703-4.

Al-Kāshī's Contributions

The methodology of this research sufficiently explained its main goal which is to investigate the scientific contributions of al-Kāshī and other Muslim scientists by following a critical, analytical and argumentative line of investigation. This means that every single scientific contribution attributed to these scientists will be rigorously examined regarding their respective roles in the establishment, initiation or development of the contribution in question. This section will investigate the contributions of al-Kāshī and other scientists in terms of initiation and development.

The Samarkand Observatory

The Samarkand Observatory is generally known as Ulugh Beg's Observatory after Ulugh Beg (1394-1449) the grandson of Tamerlane and the ruler of Transoxiana, a region in Northeast Persia in current day Uzbekistan. Ulugh Beg made Samarkand his capital. Work in the observatory was ordered in the 1420s by Sultan Ulugh Beg who was also an astronomer. Ulugh Beg ascended the throne of Transoxiana in 1447 upon the death of Shah Rukh, his father but his rule was of short duration as it was terminated two years later when he was assassinated by a killer hired by his son Abdul Latif. The observatory did not endure for a long time after 1449 and was neglected for a period of about four centuries till it was rediscovered in 1908.

Ulugh Beg invited skilful astronomers and mathematicians to build the observatory and teach in the madrasa that he established in Samarkand. Among those he invited were Ghiyāth al-Dīn Jamshīd al-Kāshī, Muin al-Dīn al-Kāshī, Salah al-Dīn Qade Zade al-Rūmi, and Ali Qūshji. To build the observatory more than 60 astronomers and mathematicians were called among whom Jamshīd al-Kāshī was the first director of the observatory.[47]

Ulugh Beg had passion in astronomy since his early youth, and his original intent was to provide Samarkand with an observatory and will make it a place for astronomers and mathematician to work together and make new findings, and exchange views and advance mathematics and astronomy. The

[47] Yazdi, H. G. (2015). 'Chronology of the Events of the Samarkand Observatory and School Based on Some Old Persian Texts: a Revision', *Suhayl*, pp.145-65.

original plan was to make a replica of the Meragha Observatory, which was built under the guidance of the famous astronomer Nasir al-Dīn al-Tūsī (1201-74). Ulugh Beg himself visited the remains of Meragha Observatory in order to get insight about the design of his own observatory. Ulugh Beg dedicated his entire life to the observatory, especially because his father was there to spare him the burden of administering the country throughout most of the last days of Ulugh Beg's life. This left Ulugh Beg with ample time to pursue the scholarly matters of the nation, among which the observatory and the madrasa were the most important. At the Meragha Observatory the principal material for astronomical discoveries was the il-Khānī Zīj (Il-Khani Astronomical Tables) which were developed by Nasir al-Dīn al-Tūsī himself, and to support the Samarkand Observatory, al-Kāshī developed the al-Khaqānī Zīj (Al-Khaqani Astronomical Tables).[48]

Among the major sources that speak about the role of al-Kāshī in the observatory construction and the supervision of the activities that were performed thereafter, is the letter that al-Kāshī sent to his father between 1421 and 1422. In this letter al-Kāshī tells us that construction works started in the observatory in 1420. Based on this letter, between 60 and 70 astronomers took part in the construction activities, but only four of them beside al-Kāshī were known to the sources.[49]

Al-Kāshī, in the letter also told his father that Ulegh Beg offered him the post of leading the other astronomers in the construction works of the observatory, and based on his advice Ulugh Beg ordered a giant meridian arch or a “geometrical platform” be built which then took the name of Suds-i-Fahri[50]. It took only three years to wrap up the construction of both the observatory and the platform, and by 1423 the giant structure was ready for astronomical work.[51]

The contents of al-Kāshī’s letter to his father represent clear evidence that al-Kāshī had a major role in building the

[48] Alain, J. (2007). ‘Prince of Samarkand Stars’. *The Mathematical Tourist*, pp.44-50.
[49]Sayili, A. (1960a). ‘Giyath al-Dīn al-Kāshī's Letter on Ulugh Beg and the Scientific Activity in Samarkand’. *Ankara: Turkish History Society*, p.106.
[50] Sayili, A. (1960b). ‘The Observatory in Islam and its Place in the General History of the Observatory’. *New York: Arno Press*,p.51.
[51]*Ibid*., p.11.

observatory, but other sources indicate that Ulugh Beg was always there to supervise all the construction activities and the astronomical observations in the observatory after that till his unfortunate assassination in 1449. Furthermore, al-Kāshī himself died in 1429 and hence whatever role he may have had in the observatory was definitely short-lived. However, two great accomplishments that have to do with the observatory may fairly be ascribed to al-Kāshī: the development of al-Khaqānī Zījor (Al-Khaqani Astronomical Tables) and the construction of Suds-i-Fahri, the giant meridian arch.[52] But while in Samarkand working on the observatory, and providing it with advanced equipment and astronomical tables, al-Kāshī relentlessly continued with his astronomical and mathematical works, thus Samarkand and the observatory provided al-Kāshī with the best environment in which to perform his genius scientific feats.[53]

Kennedy[54] (1960b) indicated that al-Kāshī himself highly prized Ulugh Beg in the letter, not only for his smooth running of the construction works, but also for his great mathematical knowledge, especially his aptitude to solve difficult mathematical problems by performing intricate mental computations. This puts Ulugh Beg well above al-Kāshī regarding the daily operations of the observatory, but al-Kāshī may also be regarded as the chief astronomer upon whom Ulugh Beg may have relied in many tasks including al-Khaqānī Zīj and the Suds-i-Fahri. Nevertheless, those sources which assumed that al-Kāshī was always at the top of the activities in the observatory may reasonably be considered inaccurate. Furthermore, Ulugh Beg survived al-Kāshī who died in 1427 and Qaḍe Zade succeeded him as the head of astronomers. Al-Kāshī also spoke with much disdain about Ulugh Beg's appointment of sixty astronomers to collaborate in the construction works, but he still praised Qade Zade as the most qualified and learned among the scientist collaborators.[55]

[52] Alain, J. (2007). 'Prince of Samarkand Stars'. *The Mathematical Tourist*, pp.44-50.
[53]Bartold, V. V. (1963). 'Four Studies on the History of Central Asia'. Vol. II, translated by V. Minorsky, Leiden, Netherlands: *Brill Academic Publishers,* p.21.
[54]Kennedy, E. S. (1960b). 'The Planetary Equatorium of Jamshīd Ghiyāth al-Dīn al-Kāshī'. Princeton, New Jersey: *Princeton University Press,* p.130.
[55]*Ibid*, p.130.

Elliptic Movement of Planets

Among al-Kāshī's contributions to astronomy while in Samarkand is the invention of an instrument that he called the "Plate of Zones" which can carry out the functions of a mechanical planetary computer that can graphically solve several planetary problems including the true locations of the celestial bodies of the solar system including the sun, the moon and the planets along with their paths while moving in their elliptical orbits[56]. The Plate of Zones also had a ruler and an alidade used for astronomical mapping.[57]

Kennedy (1952) cited al-Kāshī indicating that planets move in elliptical orbits, which is supported by some Muslim sources that claimed that al-Kāshī discovered the elliptical orbits of the moon and Mercury. But while the moon is a satellite of the earth, Mercury revolves around the sun. This claim by Kennedy (1952) was interpreted as evidence that al-Kāshī was aware of the elliptical orbits of planets. However, this source did not expressly indicate that al-Kāshī was aware that the planets of the solar system rotate around the sun.

The problem which should be investigated here is whether al-Kāshī advocated the heliocentric notion (with the sun being the universe centre) or the geocentric one (with the earth being the universe centre). However, one source, namely a colleague of al-Kāshī called Ali Qūshji, who worked on the observatory during al-Kāshī years, claims that al-Kāshī advocated the general heliocentric theory. After the death of Ulugh Beg and the termination of the astronomical operations at the observatory, Ali Qūshji went to Istanbul to set up an observatory there. According to Ragep (2005)[58] Ali Qūshji's writings about the heliocentric theory may well have influenced Copernicus, providing him with enough information to put forward his own heliocentric theory. Many scientists see

[56] Kennedy, E. S. (1950). 'A Fifteenth-Century Planetary Computer: Al-Kāshī's 'Tabaq Al-Manateq' I. Motion of the Sun and Moon in Longitude'. *Isis*, 41(2), 180-183.
[57] Kennedy, E. S. (1952). 'A Fifteenth-Century Planetary Computer: Al-Kāshī's 'Tabaq Al-Maneteq' II: Longitudes, Distances, and Equations of the Planets'. *Isis*, 43(1), 42-50.
[58] Ragep, F. J. (2005). 'Ali Qushji and Regiomontanus: Eccentric Transformations and Copernican Revolutions'. *Journal History of Astronomy*, 36, 359-371.

the argument advanced by Ragep[59] (2005) in this respect as very convincing and is now accepted by many.

Decimal Fractions

Among the great accomplishments that al-Kāshī made in mathematics was his use of decimal fractions, which influenced not only the progress of pure mathematics but also applied mathematics, physics and many other disciplines that use mathematical calculations. Many scientists of his time and European scientists thereafter recognized the great achievement made by al-Kāshī in decimal fractions.

Professor David Eugene Smith indicated in Volume II of his book "History of Mathematics" that “mathematicians differed in many things, but most of them agreed that it was al-Kāshī who invented decimals”[60]. Dr. Dirk Struik indicated in his book "Mathematics Sources" that al-Kāshī might be the sole introducer of the decimal fractions, and this can be seen clearly in “Miftah al-Hisab” (Key to Arithmetic) in which al-Kāshī used decimals very often to solve mathematical problems. In discussing decimal fractions, Struik (1986) states that:

Decimal fractions have been used in mathematical calculations by the Chinese for many centuries before Stevin to whom Europeans attribute their first use. Al-Kāshī, a Persian astronomer, first used sexagesimal fractions and then decimal fractions to solve mathematical problems in his book “Key to Arithmetic” in Samarkand in the early fifteenth century”.[61]

Although al-Kāshī made great accomplishments on decimal fractions, the above statements made by this great mathematician cannot be accepted at face value. First, one should be reminded that many ancient cultures used to calculate using numerals based on ten, perhaps because the human hands have ten fingers [62] . For instance, many civilizations during the period (3300-1300 BC) used standardized ratios of weights and lengths based on the

[59] *Ibid.,* pp.359-371
[60] Smith, D. E. (1925). ‘Hisory of Mathematics’ II, New York: Ginn and Company, p.593.
[61] Struik, D. J. (1986). ‘A Source Book in Mathematics 1200-1800’, Princeton, New Jersey: *Princeton University Press,* p.527.
[62] Dantzig, T. (1954). ‘Number: The Language of Science’. (4th ed.). London: Macmillan Publishing Co., p.12.

numerals of 10, 50, 100, and 200[63]. Some Egyptian hieroglyphs show that purely decimal systems have been used by ancient Egyptians perhaps since around 3000 BC, while Cretan hieroglyphs (c. 1625−1500 BC) used numerals that resemble the Egyptian model.[64]

In fact, as indicated earlier in this research, al-Kāshī devised the approach now known as Horner's method, in which he used decimal factions, which his predecessors may have borrowed from the Chinese.[65]

As we have shown in this research that the earlier Chinese and Indian mathematicians used some versions of decimals in the mathematical calculations they used to carry out, the credit that may go to al-Kāshī is his genius methodical use of decimal fractions by introducing a method where decimal fractions would replace the sexagesimal fractions in all calculations and thereby he created an arithmetic system to carry out these calculations.

Approximation of Pi

One of al-Kāshī's greatest achievements is the calculation of π to 9 accurate sexagesimal places or 16 decimal places. He calculated this value as part of his "Treatise on the Circumference" in 1424. Hogendijk (2009) regarded this feat as among the greatest accomplishments ever achieved by any medieval Islamic mathematician. The early Greek decimal approximations made by Archimedes and Ptolemy reached a sexagesimal value of π that is equivalent to two decimal places in the decimal system, and approximation of π made by al-Kāshī's Muslim predecessors was not much better. However, the Chinese scientist Tsu Chung-Chih, of the 5th century, managed to approximate π accurately to six decimal places. The first European scientist to approximate π to a value closer

[63] Coppa, A. et al. (2006). 'Early Neolithic Tradition of Dentistry', *Nature*, 440(7085), 56-755. Bisht, R. S. (1982). 'Excavations at Banawali: 1974-77'. New Delhi: *Oxford and IBH Publishing Co*., pp.113-24

[64] Graham, F. (2002). 'Numbers: their History and Meaning'. *Courier Dover Publications*, 50. Georges, I. (1988). 'From One to Zero: A Universal History of Numbers'. Bristol: *Penguin Books*, pp.213-18.

[65] Merzbach, U. C. and Boyer, C. B. (2011). 'A History of Mathematics'. 3rd Edition, Hoboken, New Jersey: *John Wiley and Sons Inc,* p.222.

to that of al-Kāshī was Adrian van Roomen who calculated π in 1597 to fifteen decimal places, i.e. it took the Europeans 173 years to reach a result comparable to that of al-Kāshī. However, Ludolf van Ceulen calculated π in 1615 to twenty decimal places and later to thirty-two.[66]

Al-Kāshī's objective behind the accurate approximation of π at that time was to settle an argument regarding the circumference of the earth, of which he righteously claimed that he will make an approximation with an error as little as the width of a horse's hair. For this highly accurate approximation, al-Kāshī used a polygon with $3x2^{28}$ = 805,306,368 sides. Since this great accomplishment of al-Kāshī was not available to the Europeans in English until recently, many European writers produced incorrect statements about the history of π. For instance, no mention of al-Kāshī was made in "A History of Pi", a popular book about this subject.[67]

Approximation of Sin 1

As a tool by which to create astronomical tables required for the observatory, al-Kāshī calculated sin 1 in two steps; first he created his renowned cubic equation by applying Ptolemy's theorem to a quadrilateral inscribed in a circle, and then he devised a clever and speedy iteration process to approximate sin 1 up to 16 accurate decimal digits (nine accurate sexagesimal places) using the root of a cubic equation.[68]

In fact al-Kāshī first developed "al-Khaqānī Zīj" (Al-Khaqani astronomical tables) by enhancing al-Tūsī's "il-Khānī Zīj" (Il-Khani astronomical tables). These astronomical tables required that he calculates sin (1^o) very accurately as he did in his calculation of π.[69]

For this calculation, al-Kāshī's first used the identity:

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[66] Hogendijk, J. P. (2009). 'Al-Kāshī's Determination of π to 16 Decimals in an Old Manuscript'. *Journal of the History of Arabic-Islamic Sciences*, 18, 73-152.
[67] *Ibid*., pp.73-152
[68] Azarian, M. K. (2004). 'Al-Kāshī's Fundamental Theorem'. *International Journal of Pure and Applied Mathematics*, 14(4), 499-509.
[69] Azarian, M. K. (1998). 'A Summary of the Mathematical Works of Giyath al-Dīn Jamshīd al-Kāshī'. *Journal of Recreational Mathematics*, 29(1), 32-42.

$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

Which he then transformed it into the cubic equation:

$x = a + bx^3$.

He then employed an algorithm to get successive approximations of sin (1^o) making the approximation more and more accurate every time he progressed in the algorithm, which is an approach still used in today's mathematics.

Al-Kāshī passed away on 22 June, 1429 before finishing the "Treatise on the Chord and the Sine" while he was working on the observatory on the outskirts of Samarkand. The manuscript written by al-Kāshī's was never found and has been lost forever (Kennedy, 1960b, pp. 6-7). But as the main objective of this treatise was to calculate sin 1, some of al-Kāshī's contemporary and successor scientists authored works in both Arabic and Persian about the sin 1 calculation driven by al-Kāshī's iteration approach. Al-Kāshī used both algebra and geometry in the approximation of sin 1 to any required accuracy which is considered by many of al-Kāshī's successors as an amazing feat. This wasn't only the most fascinating and original approximation technique, but was also the first approximation technique of its kind in the annals of mathematics and the pinnacle of mediaeval algebra. Additionally, at that time, this was the closest approximation for sin 1.[70]

Two earlier Muslim scientists, namely: Abūl Wafa al-Būzjānī (940-998), and Abūl Hasan Ibn Yūnūs (950-1009), discovered sometime in the 900s the best earlier approximations of sin 1 that were accurate to only four sexagesimal places. Al-Kāshī's determination of sin 1 in this manner wasn't outdone until Taqī al-Dīn Mohammed al-Asadī (1526-1585) offered a better result in the sixteenth century.

Al-Kāshī also created several trigonometric identities, including the identities that are called in our time the rules of sins and cosines.[71]

[70] Azarian, M. K. (2019). 'An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)'. *Mathematics Interdisciplinary Research*, 4, 11-19.
[71] Azarian, M. K. (2000). 'Miftah al-Hisab: A Summary'. *Missouri Journal of Mathematical Sciences*, 12(2), 75-95.

Extraction of Roots

In 1585, i.e. roughly a century and half after al-Kāshī, Simon Stevin (1548–1620), a Flemish scientist published a mathematical book in French with a section devoted to the extraction of roots. Steven followed an algorithm very similar to al-Kāshī method, and like al-Kāshī he presented the procedure of extracting the roots of integers and fractions to any required high degree of precision.

When we compare the extraction of roots followed by al-Kāshī and Stevin we note that both mathematicians have employed identical iterative processes which are almost similar to the ones now used in elementary schools to teach pupils how to extract square roots. Although, both scientists created their algorithms in steps which are somewhat different, their overall concept was the same as that presented in the section of al-Kāshī's works in this research.

As al-Kāshī discovered this algorithm of root extraction 150 before Stevin, it seems very probable that al-Kāshī's algorithm or a comparable root extraction algorithm created by another Muslim mathematician reached the part of Europe where Steven would be living many years before he was born.[72]

Binomial Theorem

Al-Kāshī expanded $(a + b)^n$ using an approach very similar to Newton's expansion, and used a triangle of numerals to get its coefficients, which is nowadays called Pascal triangle. According to Qurbani (1971), Omar al-Khayyām, a Persian mathematician and astronomer who is also famous for his great poems that are known as "Ruba'iyāt al-Khayyām", discovered both the binomial expansion and Pascal triangle roughly six centuries prior to the time of Pascal and Newton. However, it is well-known that the Persian scientist Abu Bakr al-Karāji (953-1029), along with several Chinese and possibly some Indians knew how to carry out the binomial expansion and the Pascal triangle before Omar al-Khayyām. Al-Kāshī admitted that some of his predecessors, possibly Omar al-

[72] Dakhel, A. (1960). 'The Extraction of the n-th Root in the Sexagesimal Notation: A Study of Treatise 3 of Miftah al-Hisab', Edited by W. A. Hijab, and E. S. Kennedy, Beirut: *American University of Beirut.* Rashed, R. (1994). 'The Development of Arabic Mathematics between Arithmetic and Algebra', (Trans: Armstrong, A.F.W.), Dordrecht: *Kluwer*, p.35.

Khayyām (1048-1131) and Nasir al-Dīn al-Tūsī (1201-1274), were the ones who first knew about the binomial theorem and the Pascal triangle that al-Kāshī employed in his expansion of $(a + b)^n$.[73]

Conclusion

We have shown in this article that Ghiyāth al-Dīn Jamshīd al-Kāshī was among the greatest astronomers and mathematicians in the annals of the Persian history and also one of the greatest scientists in Muslim history and the medieval history of the whole world. He was renowned for his amazing aptitude at solving puzzles that required complex mental calculations.

In "al-Rīsalah al-Mūhītiyah" (The Treatise on the Circumference), al-Kāshī calculated π up to 16 decimal places, which was only surpassed after 173 years by Ludolf van Ceulen who calculated π in 1615 to twenty decimal places.

"Rīsalat al-Witr wal-Jaib" (Treatise on the Chord and the Sine) may be the most brilliant work accomplished by al-Kāshī because of its vast applications. He managed in this treatise to approximate sin 1 up to 16 decimal places, an accomplishment that was unmatched until Taqi al-Dīn Mohammed al-Asadi (1526-1585) offered a better result in the 16^{th} century.

Among al-Kāshī's greatest works is "Miftah al-Hisab" (Key to Arithmetic) or (The Calculators' Key). In one part of "Miftah al-Hisab" al-Kāshī used the approaches of numerous Muslim mathematicians to extract integer roots with high precision.

Although some sources credited al-Kāshī with the discovery of decimal fractions in his arithmetic computations of Miftah al-Hisab, other sources found that the decimal fractions were known to many scientists before the time of al-Kāshī as earlier Chinese and Indian mathematicians used them many centuries ago, but al-Kāshī might be the first to use the decimal fractions in a methodical way. Furthermore, al-Kāshī provided a general rule for the expansion of $(a + b)^n$ that was very similar to

[73] Azarian, M. K. (2019). 'An Overview of Mathematical Contributions of Al-Kāshī (Kāshāni)'. *Mathematics Interdisciplinary Research*, 4, 11-19.

Newton's expansion, and he used a triangular table similar to Pascal triangle to get the coefficients.

Thus, we have extensively covered in this research the works and contributions of Jamshīd al-Kāshī to both mathematics and astronomy. Although we have refuted in this article many claims by some sources which falsely attributed some feats to al-Kāshī in which he was not the first to introduce, invent, or initiate, we have provided evidence that shows that al-Kāshī was indeed a great scientist. He was certainly an astronomer and a mathematician with exceptional capabilities, and an observer with a keen eye for detail, a resourceful inventor, a brilliant innovator, and a productive author.

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