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# Fifth And Eighth Order Iterative Methods For Solving Nonlinear Equations With Their Basins Of Attraction

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Abstract

In this paper, we propose and discuss multi point iterative methods for solving nonlinear equations. These methods are based on classical newton method and have fifth and eighth order convergence. The convergence analysis of proposed methods have proved. The effectiveness and performance of the new iterative methods have been tested and compared with few existing equivalent methods on various numerical examples. Finally, the basins of attraction for various polynomials are demonstrated to display the stability of method with respectto the initial point.

Keywords: Nonlinear equations; Newton method; Order of convergence; Higher order; Efficiency index; Basins of attraction

## 1. Introduction

In diverse fields of engineering and mathematical science, searching an analytic solution of nonlinear equation of the form $f(x) = 0$; $f : D \subseteq R \rightarrow R$ is not always effortless and feasible to some extent. Therefore, in order to obtain approximate solution, the numerical perspective, is not only indispensable but also extensively used inmany real word problems.

One of the most known one-step optimal quadratically convergent iterative scheme for the solution of such nonlinear equation is classical Newton method (see [1, 2]), stated by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

In recent years, a number of researchers have regularly improved newton method to propose and analyze higher order iterative methods in order to achieve more accurate results such as Chebyshev method, Halley method, Jarrat method, King's method etc. When order of convergence of iterative method increases then the number offunctional evaluations per step also increases. The Ostrowski index (see[3]) gives a measure of balance betweensuch quantities, according to the formula $E= p^{(1/d)}$, where p is the order of convergence of the method and d is thetotal number of function evaluations per step. Recently, Many fifth and eighth order iterative methods have beendeveloped in the literature(see[4–15] and references therein). The recent publication by Petkovic[16] comprises a survey on the developments that have been made in the class of multi-point methods as well as an extended list of references.

In this study, we contribute a little bit more in the theory of iterative procedures by proposing iterative procedures which are free from second derivatives and are of fifth and eighth orderof convergence. The rest of this paper is organized as follows: The proposed methods are described in Section 2. The convergence analysis is accomplish to demonstrate the order of convergence in Section 3. Comparison of presented methods with various existing methods of fifth and eighth order in series of numerical illustrations is presented in Section 4. Section 5 is devoted to the complex dynamical analysis of the designed methods on quadratic, cubic, bi-quadratic and fifth degree polynomials. In Section 6, we end this paper with some concluding remarks.

## 2. The Proposed Methods

Assume that r is a simple zero of f (x); i.e. f (r) = 0, $f'(r) = 0$ and f (x) is a real function which is sufficientlydifferentiable on an interval.

**Algorithm 1:-** For initial approximation $x_0$, the approximate solution $x_{k+1}$ of $f(x) = 0$ can be computed by thefollowing iterative method:

$$y_K = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_k = x_k - \frac{1}{8}\left[\frac{1}{f'(x_k)} + \frac{3}{f'(y_k)}\right] f(x_k),$$
$$x_{k+1} = x_k - \frac{6f(x_k)}{f'(x_k) + f'(y_k) + 4f'(z_k)}.$$

(2)

The order of convergence of above method is five. It requires one evaluation of f and three evaluations of $f'$ in each iterations. We call this iterative method as $AR_5$.

**Algorithm 2:-** For a given initial approximation $x_0$, computing the approximate solution $x_{k+1}$ of $f(x) = 0$ by thefollowing iterative scheme:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_k = x_k - \frac{1}{8}\left[\frac{1}{f'(x_k)} + \frac{3}{f'(y_k)}\right] f(x_k),$$
$$w_k = x_k - \frac{6f(x_k)}{f'(x_k) + f'(y_k) + 4f'(z_k)},$$
$$x_{k+1} = w_k - \left[\alpha + \left[\beta\frac{f'(y_k)}{f'(x_k)} + \gamma\frac{f'(z_k)}{f'(x_k)} + \delta\right]\frac{f'(z_k)}{f'(x_k)}\right]\frac{f(w_k)}{f'(x_k)}.$$

(3)

The order of convergence of the above scheme is eight. It requires two evaluations of f and three evaluations of

$f'$ in each iterations. We designated this method as $AR_8$.

**Remark: -** One additional step is added in fifth order method defined in Algorithm 1 to achieve the convergence order eight defined in Algorithm 2.

## 3. Convergence Analysis

Here, we have proved the convergence of above proposed iterative schemes with the help of MATHEMATICA

13.0 software.

**Theorem 1** . *Let* $f : D \subseteq \mathbb{R} \to \mathbb{R}$, be a real valued function defined on $D$, where $D$ is a neighborhood of a simple zero $r$ of $f(x)$. Suppose that $f(x)$ is sufficiently smooth in $D$ and differentiable in the neighborhood of $r$, then the iterative scheme defined by (2) has order of convergence at least five.

**Proof.** Let $e_k = x_k - r$, $\tilde{e}_k = y_k - r$, $\hat{e}_k = z_k - r$.
Using Taylor's series expansion of $f(x_k)$ and $f'(x_k)$ about $r$ and assuming that $f(r) = 0, f'(r) \neq 0$

$$f(x_k) = f'(r)\left[e_k + A_2e_k^2 + A_3e_k^3 + A_4e_k^4 + A_5e_k^5 + A_6e_k^6 + O(e_k^7)\right], \tag{4}$$

$$f'(x_k) = f'(r)\left[1 + 2A_2e_k + 3A_3e_k^2 + 4A_4e_k^3 + 5A_5e_k^4 + 6A_6e_k^5 + O(e_k^6)\right], \tag{5}$$

where $A_i = \frac{1}{i!}f^i(r)$,
From (4) and (5), we get

$$\frac{f(x_k)}{f'(x_k)} = e_k - A_2e_k^2 + 2(A_2^2 - A_3)e_k^3 + (-4A_2^3 + 7A_2A_3 - 3A_4)e_k^4 + B_1e_k^5 + B_2e_k^6 + B_3e_k^7 + O(e_k^8). \tag{6}$$

following are the expressions of $B_n(n = 1, 2, 3)$

$$B_1 = 8A_2^4 - 20A_2^2A_3 + 6A_3^2 + 10A_2A_4 - 4A_5,\ B_2 = -16A_2^5 + 52A_2^3A_3 - 33A_2A_3^2 - 28A_2^2A_4 + 17A_3A_4 + 13A_2A_5 - 5A_6,$$
$$B_3 = 32A_2^6 - 128A_2^4A_3 + 63A_2^2A_3^2 - 18A_3^3 + 72A_2^3A_4 - 46A_2A_3A4 + 6A_4^2 - 18A_2^2A_5 + 11A_3A_5 + 8A_2A_6 - 3A_7.$$

Using (6) in the first substep of the scheme (2), we get

$$\tilde{e}_k = A_2e_k^2 - 2(A_2^2 - A_3)e_k^3 + (4A_2^3 - 7A_2A_3 + 3A_4)e_k^4 - B_1e_k^5 - B_2e_k^6 - B_3e_k^7 + O(e_k^8). \tag{7}$$

Now, the following expansion of $f'(y_k)$ about $r$, is obtained by using the above result

$$f'(y_k) = f'(r)[1 + 2A_2\tilde{e}_k + 3A_3\tilde{e}_k^2 + 4A_4\tilde{e}_k^3 + O(\tilde{e}_k^4)] = f'(r)[1 + 2A_2^2e_k^2 - 4(A_2^3 - A_2A_3)e_k^4 + C_1e_k^4 + C_2e_k^5 + O(e_k^6)]. \tag{8}$$

where

$$C_1 = 8A_2(A_2^3 - 11A_2A_3 + 6A_4),\ C_2 = -4A_2(4A_2^4 - 7A_2^2A_3 + 5A_2A_4 - 2A_5).$$

From (4) and (8), It follows that

$$\frac{f(x_k)}{f'(y_k)} = e_k + A_2e_k^2 - (2A_2^2 - A_3)e_k^3 + (2A_2^3 - 4A_2A_3 + A_4)e_k^4 + (5A_2^2A_3 - 6A_2A_4 + A_5)e_k^5 + O(e_k^6). \tag{9}$$

Substituting (6) and (9) in second sub-step of (2), we have

$$\hat{e}_k = \frac{1}{2}e_k - \frac{}{A_2}4e_k^2 + \frac{1}{8}(4A_2^2 - A_3)e_k^3 + O(e_k^4). \tag{10}$$

Expanding $f'(z_k)$ about $r$, we obtain

$$f'(z_k) = f'(r)[1 + 2A_2\hat{e}_k + 3A_3\hat{e}_k^2 + 4A_4\hat{e}_k^3 + O(\hat{e}_k^4)] = f'(r)[1 + A_2e_k + \frac{1}{4}(-2A_2^2 + 3A_3)e_k^2 + \frac{1}{2}(2A_2^3 - 2A_2A_3 + A_4)e_k^3 + O(e_k^4)]. \tag{11}$$

By invocation of (4), (5), (8) and (11) in the last step of (2), we get

$$e_{k+1}=\frac{1}{24}(24A_2^4+3A_2^2A_3-6A_3^2+12A_2A_4+A_5)e_k^5-\frac{1}{24}(120A_2^5-105A_2^3A_3+30A_2^2A_4+6A_3A_4-5A_2(3A_3^2+5A_5)-3A_6)e_k^6+O(e_k^7). \quad (12)$$

Hence, the fifth order convergence is established. Which completes the proof of the theorem.

**Theorem 2** . *Let* $f: D\subseteq\mathbb{R}\rightarrow\mathbb{R}$, be a real valued function defined on $D$, where $D$ is a neighborhood of a simple zero $r$ of $f(x)$. Suppose that $f(x)$ is sufficiently smooth in $D$ and differentiable in the neighborhood of $r$, then the iterative scheme defined by (3) has order of convergence at least eight provided $\alpha=9,\ \beta=-1,\ \gamma=8,\ \delta=-15$.

**Proof.** Let $\ddot{e}_k=w_k-r$. Renaming the equation (12), we get

$$\ddot{e}_k=\frac{1}{24}(24A_2^4+3A_2^2A_3-6A_3^2+12A_2A_4+A_5)e_k^5-\frac{1}{24}(120A_2^5-105A_2^3A_3+30A_2^2A_4+6A_3A_4-5A_2(3A_3^2+5A_5)-3A_6)e_k^6+O(e_k^7). \quad (13)$$

Expanding $f(w_k)$ about $r$, we have

$$f(w_k)=f'(r)\left[\ddot{e}_k+A_2\ddot{e}_k^2+A_3\ddot{e}_k^3+A_4\ddot{e}_k^4+A_5\ddot{e}_k^5+A_6\ddot{e}_k^6+O(\ddot{e}_k^7)\right]=f'(r)[E_1e_k^5+E_2e_k^6+O(e_k^7)]. \quad (14)$$

Where

$$E_1=\frac{1}{24}(24A_2^4+3A_2^2A_3-6A_3^2+12A_2A_4+A_5),\ E_2=-\frac{1}{24}(120A_2^5-105A_2^3A_3+30A_2^2A_4+6A_3A_4-5A_2(3A_3^2+5A_5)-3A_6).$$

Substitueing (5), (8), (11), (12) and (14) in the last sub step of proposed scheme (3), we have

$$e_{k+1}=L_1e_k^5+L_2e_k^6+L_3e_k^7+L_4e_k^8+O(e_k^9).$$

Where

$$
\begin{aligned}
M_1 &= \alpha+\beta+\gamma+\delta-1, \\
L_1 &= -\frac{1}{24}M_1(24A_2^4+3A_3A_2^2+12A_4A_2-6A_3^2+A_5), \\
L_2 &= \frac{1}{24}(24A_2^5(7\alpha+10\beta+9\gamma+8\delta-5)-3A_3A_2^3(33\alpha+30\beta+31\gamma+32\delta-35)+6A_4A_2^2(9\alpha+15\beta+13\gamma+11\delta-5) \\
&- 3A_2A_3^2(9\alpha+15\beta+13\gamma+11\delta-5)+A_2A_5(23\alpha+20\beta+21\gamma+22\delta-25)+3(2A_3A_4-A_6)M_1), \\
L_3 &= \frac{1}{96}(-48A_2^6(58\alpha+119\beta+94\gamma+75\delta-30)+6A_3A_2^4(630\alpha+893\beta+826\gamma+729\delta-450)-12A_4A_2^3(103\alpha+195\beta \\
&+ 155\gamma+127\delta-67)+A_2^2\left(2A_5(207\alpha+326\beta+291\gamma+250\delta-115)-3A_3^2(144\alpha-61\beta+22\gamma+87\delta-228)\right) \\
&+ 6A_2\left(2A_3A_4(26\alpha+41\beta+40\gamma+33\delta-18)-A_6(21\alpha+15\beta+17\gamma+19\delta-25)\right)-3A_3^3(49\alpha+91\beta+85\gamma+67\delta-25) \\
&+ 3A_3A_5(4\alpha+11\beta+10\gamma+7\delta)-23A_7M_1), \\
L_4 &= \frac{1}{96}(48A_2^7(186\alpha+547\beta+372\gamma+269\delta-70)-6A_3A_2^5(3206\alpha+6737\beta+5500\gamma+4287\delta-1610)+12A_4A_2^4(587\alpha \\
&+ 1235\beta+949\gamma+753\delta-349)+A_2^3(A_3^2(7860\alpha+9201\beta+9696\gamma+8895\delta-5808)-4A_5(462\alpha+964\beta+754\gamma \\
&+ 598\delta-255))-6A_2^2(A_3A_4(697\alpha+1229\beta+1073\gamma+869\delta-493)-2A_6(50\alpha+72\beta+67\gamma+59\delta-29)) \\
&+ A_2(3A_3^3(37\alpha+613\beta+437\gamma+219\delta+169)+A_5A_3(642\alpha+969\beta+962\gamma+811\delta-390)+24A_4^2(26\alpha+41\beta+40\gamma \\
&+ 33\delta-18)-A_7(150\alpha+81\beta+104\gamma+127\delta-196))-6A_3^2A_4(41\alpha+92\beta+87\gamma+64\delta-13)+3A_3A_6(5\alpha+26\beta \\
&+ 23\gamma+14\delta+7)-2A_4A_5(2\alpha-13\beta-12\gamma-5\delta-10)-36A_8M_1).
\end{aligned}
$$

For the method to be of order eight, we must have $\alpha=9,\ \beta=-1,\ \gamma=8,\ \delta=-15$. With these values, above equation yields

$$e_{k+1}=-\frac{1}{48}\left(A_2(2A_2^2+25A_3)(24A_2^4+3A_2^2A_3+12A_2A_4-6A_3^2+A_5)\right)e_k^8+O(e_k^9).$$

Hence, eighth order convergence is achieved. Which completes the proof of the theorem

## 4. Numerical illustrations

In order to illustrate the efficiency and performance of the proposed methods, few numerical examples are considered. We compare the proposed methods with some existing fifth and eighth order methods in literature. Note that all computations are executed using MATHEMATICA 13.0 software. We conceive an approximate solution rather than the exact root. The total number of iterations (N) are listed by using the stopping criteria either

$$|x_N-x_{N-1}|<10^{-50}\ or\ |f(x_N)|<10^{-50}.$$

When the stopping criteria is satisfied, $x_N$ is taken as computed root r. It is the well-known fact that, in case of multi-step iterative methods, a fast convergence can be achieved if initial approximation is very close to the root;for this reason, a special attention should be paid to exploring appropriate initial approximation [1]. Furthermore, any iterative method may be divergent when initial approximation is away from the exact root.

For exhibiting numerical results, some existing fifth and eighth order methods from literature are given below:

Ham et al. method ($HM5$)[4] is given by:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \frac{f(y_k)}{f'(x_k)} \frac{f'(y_k) + 3f(x_k)}{5f'(y_k) - f'(x_k)}. \tag{15}$$

Fang et al. method ($FM5$)[5] is given by:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \frac{5f'^2(x_k) + 3f'^2(y_k)}{f'^2(x_k) + 7f'^2(y_k)} \frac{f(y_k)}{f'(x_k)}. \tag{16}$$

Sharma method ($SM5$)[6] is given by:

$$y_k = x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)},$$
$$z_k = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)} - \frac{f(x_k)}{3f'(y_k) - f'(x_k)},$$
$$x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}. \tag{17}$$

Kumar et al. method ($KM5$)[7] is stated as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_{k+1} = x_k - \frac{f(x_k) + f(y_k)}{f'(x_k)},$$
$$x_{k+1} = z_{k+1} - \frac{f(z_{k+1})}{f'(y_k)}. \tag{18}$$

Zang et al. method ($ZM5$)[8] is stated as:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_k = y_k - \frac{f(x_k)}{f'(x_k)} \frac{f(y_k)}{f(x_k) - 2f(y_k)},$$
$$x_{k+1} = z_k - \frac{f(z_k)}{f'(x_k)}. \tag{19}$$

Bawazir method ($BM5$)[9] is given by:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)}\left[1 + \frac{f(y_k)(f'(x_k) - f'(y_k))}{2f(x_k)f'(y_k)}\right]. \tag{20}$$

Liu et al. method ($LM8$)[10] is stated as:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
z_k &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)},\\
x_{k+1} &= z_k - \left[\left(\frac{f(x_k)-f(y_k)}{f(x_k)-2f(y_k)}\right)^2 + \frac{f(z_k)}{f(y_k)-f(z_k)} + \frac{4f(z_k)}{f(x_k)+f(z_k)}\right]\frac{f(z_k)}{f'(x_k)}.
\end{aligned} \tag{21}
$$

Sharma et al. method ($SM8$)[11] is given by:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
z_k &= y_k - \left[3-2\frac{f[y_k,x_k]}{f'(x_k)}\right]\frac{f(y_k)}{f'(x_k)},\\
x_{k+1} &= z_k - \left[\frac{f'(x_k)-f[y_k,x_k]+f[z_k,y_k]}{2f[z_k,y_k]-f[z_k,x_k]}\right]\frac{f(z_k)}{f'(x_k)}.
\end{aligned} \tag{22}
$$

Cordero et al. method ($CM8$)[12] is stated as:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
z_k &= y_k - \frac{f^3(x_k)}{f^3(x_k)-2f^2(x_k)f(y_k)-f(x_k)f^2(y_k)-\frac{1}{2}f^3(y_k)}\frac{f(y_k)}{f'(x_k)},\\
x_{k+1} &= z_k - \frac{f(x_k)+3f(z_k)}{f(x_k)+f(z_k)}\frac{f(z_k)}{f[z_k,y_k]+f[z_k,x_k,x_k](z_k-y_k)}.
\end{aligned} \tag{23}
$$

Sharma et al. method($SNM8$)[13] is given by:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
z_k &= y_k - \left[1+6\frac{f(y_k)}{f(x_k)}+9\left(\frac{f(y_k)}{f(x_k)}\right)^2+14\left(\frac{f(y_k)}{f(x_k)}\right)^3\right]^{\frac{1}{3}}\frac{f(y_k)}{f'(x_k)},\\
x_{k+1} &= z_k - \left[2\frac{f(y_k)}{f(x_k)}+\frac{1+4\frac{f(z_k)}{f(x_k)}}{1-\frac{f(z_k)}{f(y_k)}}\right]\frac{f(z_k)}{f'(x_k)}.
\end{aligned}
$$

Lotfi et al. method ($LSM8$)[14] is given by:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
z_k &= y_k - \frac{f(y_k)}{f(x_k)-2f(y_k)}\frac{f(x_k)}{f'(x_k)},\\
x_{k+1} &= z_k - \frac{f(y_k)+\frac{2f^2(y_k)}{f(x_k)}+\frac{5f^3(y_k)}{f^2(x_k)}+\frac{12f^4(y_k)}{f^3(x_k)}+\frac{2f(y_k)f(z_k)}{f(x_k)}}{f(y_k)-f(z_k)}\frac{f(z_k)}{f'(x_k)},
\end{aligned} \tag{24}
$$

Kong method($KM8$)[15] is given by:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$
$$z_k = y_k - \frac{f^2(x_k)}{f^2(x_k) - 2f(x_k)f(y_k) + f^2(y_k)} \frac{f(y_k)}{f'(x_k)},$$
$$x_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}.$$

9(25)

Table 1: Efficiency Indices of various methods

| Methods | p | d | E |
|---|---|---|---|
| NM2 | 2 | 2 | 1.4142 |
| HM5 | 5 | 4 | 1.4953 |
| SM5 | 5 | 4 | 1.4953 |
| FM5 | 5 | 4 | 1.4953 |
| KM5 | 5 | 5 | 1.3797 |
| ZM5 | 5 | 4 | 1.4953 |
| BM5 | 5 | 4 | 1.4953 |
| AR5 | 5 | 4 | 1.4953 |
| LM8 | 8 | 4 | 1.6818 |
| PM8 | 8 | 4 | 1.6818 |
| SM8 | 8 | 4 | 1.6818 |
| CM8 | 8 | 4 | 1.6818 |
| SNM8 | 8 | 4 | 1.6818 |
| LSM8 | 8 | 4 | 1.6818 |
| KM8 | 8 | 5 | 1.5157 |
| AR8 | 8 | 5 | 1.5157 |

Table 2: Test functions with their approximate root and initial approximation:

| $f(x)$ | Root$(r)$ | Initial approximation$(x_0)$ |
|---|---|---|
| $f_1(x) = 1+e^{(2+x-x^2)} - cos(x+1) + x^3$ | -1.0000000000000000.. | -3.0 |
| $f_2(x) = \sqrt{x^2+2x+5} - 2sinx - x^2 + 3$ | 2.3319676558839637.. | 3.0 |
| $f_3(x) = ln(x^2+x+2) - x + 1$ | 4.1525907367571583.. | 2.8 |
| $f_4(x) = x^5 + x^4 + 4x^2 - 15$ | 1.3474280989683049.. | 3 |
| $f_5(x) = arctan(x) - x^2 + 1$ | 1.3961536566409308.. | 1.5 |
| $f_6(x) = sinx + cosx + x$ | -0.45662470456763082.. | -1.9 |
| $f_7(x) = x^2 + sin(\frac{x}{5}) - \frac{1}{4}$ | 0.40999201798913709.. | 0.8 |
| $f_8(x) = \sqrt{x} - cosx$ | 0.64171437087288263.. | 1.3 |
| $f_9(x) = (x-1)^3 - 1$ | 2.0000000000000000.. | 3.5 |

Table 1 shows the efficiency indices of various methods and new methods. Table 2 represents the test functions,their approximate roots r and initial approximations $x_0$.

Table 3: Comparison between various fifth order methods

| $Methods$ | $f_1(x)$ | | | $f_2(x)$ | | | $f_3(x)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ |
| $HM5$ | 10 | 7.1528(-45) | 3.8373(-89) | 7 | 4.2586(-30) | 3.2901(-60) | 5 | 8.8818(-15) | 5.3291(-67) |
| $FM5$ | 6 | 8.8249(-44) | 1.5332(-172) | 7 | 2.0597(-20) | 1.1294(-59) | 3 | 3.1757(-5) | 1.9127(-59) |
| $SM5$ | 4 | 2.9587(-10) | 1.7792(-67) | 5 | 4.4409(-16) | 8.8818(-16) | 2 | 1.5422(-3) | 2.9873(-89) |
| $KM5$ | 4 | 9.2462(-35) | 4.7927(-274) | 4 | 4.3929(-37) | 1.5671(-87) | 2 | 1.28903(-3) | 7.7630(-63) |
| $ZM5$ | 4 | 1.7033(-2) | 5.9741(-135) | 4 | 3.4045(-46) | 1.9384(-230) | 2 | 1.4547(-3) | 8.7605(-74) |
| $BM5$ | 4 | 4.6305(-20) | 1.2769(-79) | 4 | 1.3746(-42) | 1.5016(-212) | 2 | 1.2505(-3) | 7.5308(-84) |
| $AR5$ | 7 | 3.2261(-34) | 1.0408(-67) | 4 | 5.0061(-35) | 8.8327(-175) | 2 | 2.6479(-4) | 1.2466(-93) |
| $Methods$ | $f_4(x)$ | | | $f_5(x)$ | | | $f_6(x)$ | | |
| | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_n)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ |
| $HM5$ | 8 | 1.0354(-33) | 3.1642(-65) | 5 | 4.4806(-149) | 2.2205(-16) | 7 | 3.5903(-148) | 5.5511(-17) |
| $FM5$ | 7 | 3.4454(-25) | 6.8402(-72) | 4 | 3.7286(-147) | 2.2205(-16) | 5 | 4.5662(-148) | 5.5511(-17) |
| $SM5$ | 5 | 1.1136(-42) | 2.6645(-58) | 3 | 4.8759(-149) | 2.2204(-16) | 4 | 2.6196(-149) | 5.5511(-17) |
| $KM5$ | 5 | 6.4649(-39) | 2.1317(-189) | 3 | 1.6535(-146) | 2.2205(-16) | 4 | 3.2984(-141) | 5.5511(-17) |
| $ZM5$ | 4 | 7.6899(-13) | 1.1087(-59) | 3 | 9.8029(-141) | 2.2204(-16) | 4 | 3.4057(-140) | 5.5511(-17) |
| $BM5$ | 4 | 1.0519(-11) | 7.0219(-54) | 4 | 2.2205(-146) | 2.2205(-16) | 4 | 2.4599(-247) | 5.5511(-17) |
| $AR5$ | 4 | 4.1954(-12) | 6.6189(-56) | 3 | 4.7267(-147) | 2.2205(-16) | 4 | 2.2193(-242) | 5.5511(-17) |
| $Methods$ | $f_7(x)$ | | | $f_8(x)$ | | | $f_9(x)$ | | |
| | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ | $N$ | $\lvert x_N - x_{N-1}\rvert$ | $\lvert f(x_N)\rvert$ |
| $HM5$ | 5 | 2.339(-102) | 2.3845(-52) | 4 | 1.0850(-11) | 1.3267(-121) | 6 | 2.0472(-13) | 6.1417(-93) |
| $FM5$ | 4 | 1.3846(-110) | 1.4113(-31) | 3 | 3.5277(-8) | 4.3134(-83) | 5 | 7.4254(-7) | 2.2276(-60) |
| $SM5$ | 4 | 2.0550(-413) | 2.7755(-17) | 2 | 2.5591(-6) | 3.1291(-66) | 3 | 1.3268(-4) | 3.9811(-54) |
| $KM5$ | 3 | 2.584(-112) | 2.6342(-11) | 3 | 3.9215(-5) | 4.7950(-55) | 4 | 1.7763(-15) | 5.3291(-55) |
| $ZM5$ | 4 | 2.0544(-143) | 2.7755(-17) | 2 | 1.1638(-4) | 1.4230(-84) | 2 | 1.3269(-4) | 1.32687(-72) |
| $BM5$ | 4 | 2.0500(-147) | 2.7756(-17) | 3 | 1.1102(-16) | 1.11022(-68) | 2 | 1.7402(-4) | 5.2215(-104) |
| $AR5$ | 4 | 1.1102(-144) | 2.7756(-17) | 2 | 2.7052(-4) | 3.3076(-73) | 3 | 7.5977(-5) | 2.2795(-57) |

$Here\ a(-b) = a \times 10^{(-b)}$.

Table 3 represents the corresponding results for $f_1 - f_9$. The computational results clearly show that new proposed fifth order method is competitive with other existing fifth order methods. One can easily notice that there is no clear champion between the various fifth order methods in a manner that one acts well in one state while others call for other. Table 4 shows the comparable results for $f_1 - f_9$. The computational results clearly show that new proposed eighth order method is competitive with other existing eighth order methods. We can easily perceive that there is no clear titleholder between the various eighth order methods in a manner that one acts well in one state while others lead the way in other.

Table 4: Comparison between various eighth order methods

| Methods | $f_1(x)$ | | | $f_2(x)$ | | | $f_3(x)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ |
| $LM8$ | 3 | 6.4057(-12) | 1.9973(-58) | 3 | 2.6648(-15) | 4.8842(-121) | 2 | 4.0279(-5) | 2.42594(-151) |
| $SM8$ | 4 | 1.2003(-45) | 4.61410(-227) | 3 | 1.8716(-9) | 1.1713(-74) | 2 | 4.8168(-5) | 2.9011(-145) |
| $CM8$ | 5 | 1.3193(-42) | 7.4003(-212) | 3 | 5.0999(-31) | 1.9067(-248) | 2 | 7.3177(-7) | 4.4074(-171) |
| $SNM8$ | 4 | 1.2053(-35) | 4.7099(-177) | 5 | 1.3361(-42) | 4.1997(-340) | 2 | 4.8695(-5) | 2.9328(-149) |
| $LSM8$ | 4 | 5.2078(-45) | 7.0939(-224) | 3 | 9.6085(-9) | 2.9431(-69) | 2 | 1.1448(-4) | 6.8949(-205) |
| $KM8$ | 3 | 1.0958(-10) | 5.3436(-63) | 3 | 2.8963(-22) | 7.2332(-178) | 2 | 4.8170(-6) | 2.9012(-166) |
| $AR8$ | 4 | 1.1655(-21) | 5.1265(-86) | 3 | 2.1150(-13) | 1.3824(-105) | 3 | 1.0503(-5) | 6.3261(-157) |
| Methods | $f_4(x)$ | | | $f_5(x)$ | | | $f_6(x)$ | | |
| | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ |
| $LM8$ | 4 | 3.4279(-40) | 2.2871(-314) | 2 | 1.0385(-1) | 1.5229(-90) | 2 | 1.8648-2) | 5.5511(-70) |
| $SM8$ | 3 | 2.1049(-11) | 4.4311(-83) | 2 | 1.0385(-10) | 1.9842(-59) | 2 | 2.6239(-2) | 5.5511(-77) |
| $CM8$ | 4 | 5.0068(-40) | 4.9730(-313) | 2 | 1.0385(-40) | 4.7433(-110) | 2 | 1.7196(-3) | 5.5511(-71) |
| $SNM8$ | 4 | 6.8582(-26) | 7.0303(-199) | 2 | 1.0385(-1) | 5.3449(-119) | 9 | 6.6946(-16) | 5.4153(-60) |
| $LSM8$ | 4 | 9.7288(-33) | 6.0842(-254) | 2 | 1.0385(-1) | 3.3698(-69) | 3 | 5.9682(-2) | 5.8564(-64) |
| $KM8$ | 4 | 9.6909(-43) | 9.0873(-335) | 2 | 1.8909(-40) | 2.2205(-106) | 2 | 2.3241(-3) | 5.5511(-71) |
| $AR8$ | 4 | 3.8026(-40) | 2.1456(-313) | 3 | 4.4444(-47) | 2.2205(-106) | 4 | 1.0547(-45) | 5.5511(-71) |
| Methods | $f_7(x)$ | | | $f_8(x)$ | | | $f_9(x)$ | | |
| | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ | $N$ | $\|x_N - x_{N-1}\|$ | $\|f(x_N)\|$ |
| $LM8$ | 2 | 2.0271(-4) | 2.0667(-78) | 2 | 2.3033(-8) | 2.8164 (-88) | 2 | 8.4451(-2) | 2.2601(-108) |
| $SM8$ | 2 | 9.9120(-5) | 2.7756(-117) | 2 | 2.1843(-6) | 2.6708(-67) | 2 | 2.2743(-3) | 6.8385(-53) |
| $CM8$ | 2 | 1.1965(-5) | 1.2201(-96) | 2 | 7.1472(-7) | 8.7391(-71) | 2 | 5.1732(-2) | 3.1123(-101) |
| $SNM8$ | 2 | 4.2067(-4) | 4.2897(-83) | 2 | 5.2946(-7) | 6.4739(-79) | 3 | 5.4489(-7) | 1.6347(-61) |
| $LSM8$ | 2 | 2.6985(-4) | 6.8949(-76) | 2 | 2.1803(-7) | 2.6659(-78) | 3 | 5.1669(-8) | 1.5501(-77) |
| $KM8$ | 2 | 8.2522(-5) | 8.4122(-68) | 2 | 1.3126(-7) | 1.6049(72) | 3 | 5.0535(-10) | 1.5160(-89) |
| $AR8$ | 2 | 3.7526(-7) | 6.3261(-106) | 2 | 2.6822(-6) | 3.8303(-76) | 3 | 9.1686(-10) | 3.2796(-69) |

$Here\ a(-b) = a \times 10^{(-b)}$.

## 5. Basins of Attraction on the Complex Plane

In this section, we are studying the dynamical behavior of above mentioned fifth and eighth order iterative methods by generating basins of attraction for three different polynomials. Actually, basin of attraction is an approachto visually comprehend how an iterative method acts as a function of the various starting points. Stewart [17] was the first who generates the basins to compare classical newton method with various other methods of different orders. Following [18] and [14], we take a grid of 256 × 256 points in square [−2, 2] × [−2, 2] ⊆ C and [−2.5, 2.5] × [−2.5, 2.5] ⊆ C for fifth and eighth order methods resp. It contains all the roots of corresponding nonlinear equation and we use the iterative method beginning with each $z_0$ of the square. We assign each point $z_0$ a color in accordance with the simple root that the associated orbit of the iterative method, starting from $z_0$, converges. If after at most 50 iterations the orbit has a distance, to any of the roots, that is greater than $10^{-3}$, welabel the point $z_0$ as black, meaning that the orbit does not converge to a root. This is the way to differentiate thebasin of attraction by their color for various methods. Let us take four test problems

$$p_1(z) = z^2 - 1,\ p_2(z) = z^2 - \frac{1}{z},\ p_3(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4},\ p_4(z) = z^4 - \frac{1}{z}.$$

For studying convergence and stability of fifth order methods, we consider three test problems $p_1(z)$, $p_2(z)$, $p_3(z)$.

Fig 1: clearly exhibits the fractal graph of $p_1(z)$ polynomial. One can easily understand that the proposed method

AR5 seems to construct larger basins of attraction than various fifth order method HM5, FM5, SM5, KM5, ZM5 and BM5 as there is no chaotic behavior at all. Whereas SM5, KM5, ZM5 and BM5 display a little chaotic

behavior next to boundary point. HM5 and FM5 shows the smaller basin of attraction than AR5.

Fig 2: displays the fractal graph of $p_2(z)$ polynomial. We can acknowledge that the proposed method AR5 seems to construct almost competitive basins of attraction as FM5, SM5, KM5, ZM5 and BM5 as there is less chaoticbehavior nearby boundary points and larger basins of attraction than HM5.

Fig 3: exhibits the fractal graph of $p_3(z)$ polynomial. One can easily acknowledge that the proposed method AR5 seems to construct larger basins of attraction than various fifth order method HM5, FM5, SM5, KM5, ZM5 and BM5 as there is less chaotic behavior nearby boundary points.

For examining convergence and stability of eighth order methods, we take three test problems $p_1(z)$, $p_3(z)$, $p_4(z)$.

Fig 4: exhibits the fractal graph of $p_1(z)$ polynomial. We can see that the proposed method AR8 seems to construct

almost competitive basins of attraction as KM8 and larger basins of attraction than LM8, SM8, CM8, SNM8 and LSM8 as there is less chaotic behavior nearby boundary points.

Fig 5: displays the fractal graph of $p_3(z)$ polynomial. One can easily acknowledge that the proposed method AR8

seems to construct almost competitive basins of attraction as LM8, CM8, KM8 and larger basins of attractionthan SM8, SNM8 and LSM8.

Fig 6: exhibits the fractal graph of $p_4(z)$ polynomial. One can easily acknowledge that the proposed method AR8

seems to construct almost competitive basins of attraction as CM8, KM8 and larger basins of attraction than

LM8, SM8, SNM8 and LSM8.

It is clear from the figures $(1-6)$ the proposed fifth and eighth order methods show the best performance.

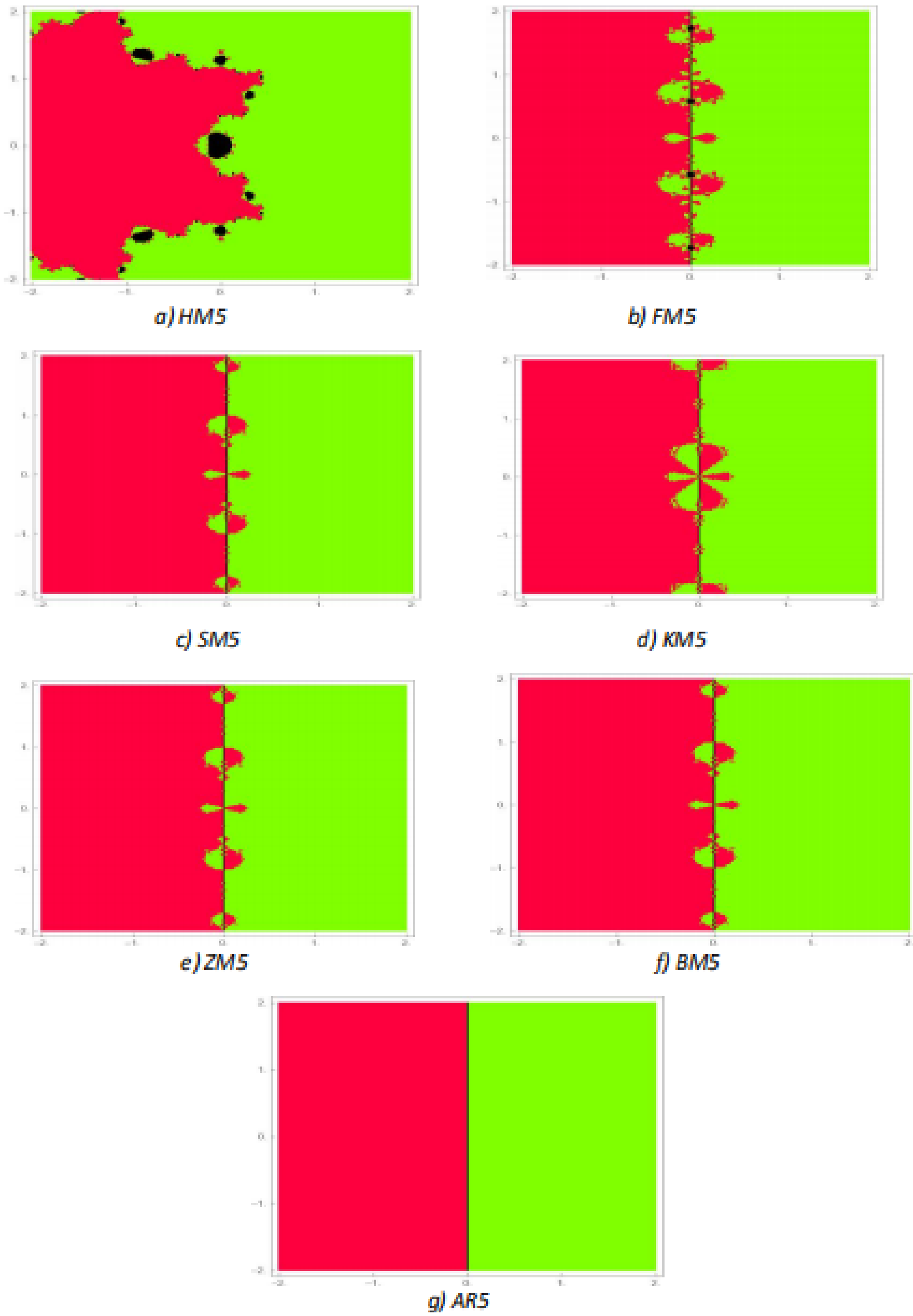



Fig 1: Basins of attraction for $p_1(z) = z^2-1$

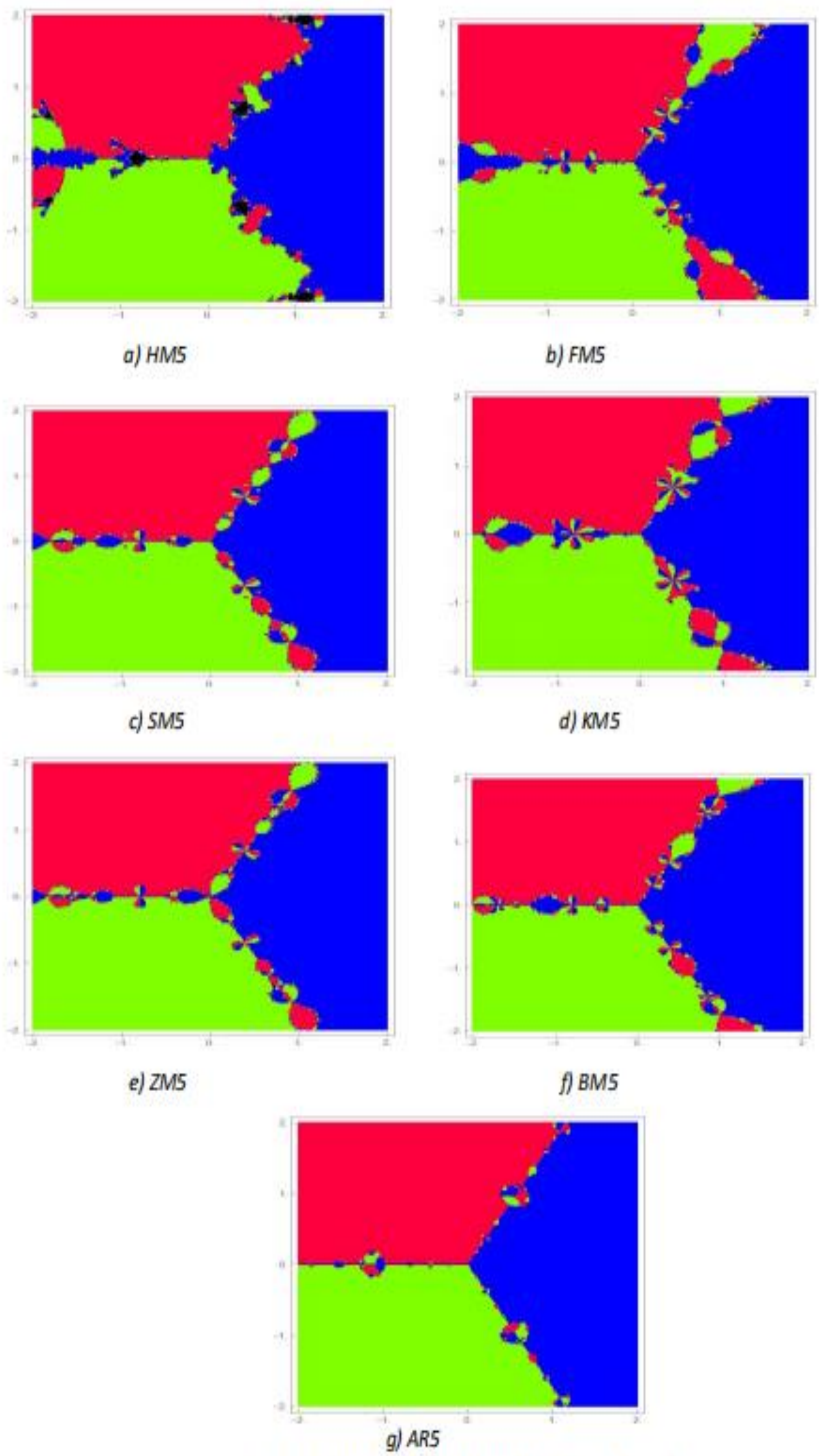



Fig 2 : Basins of attraction for $p_2(z)=z^2-1/z$.

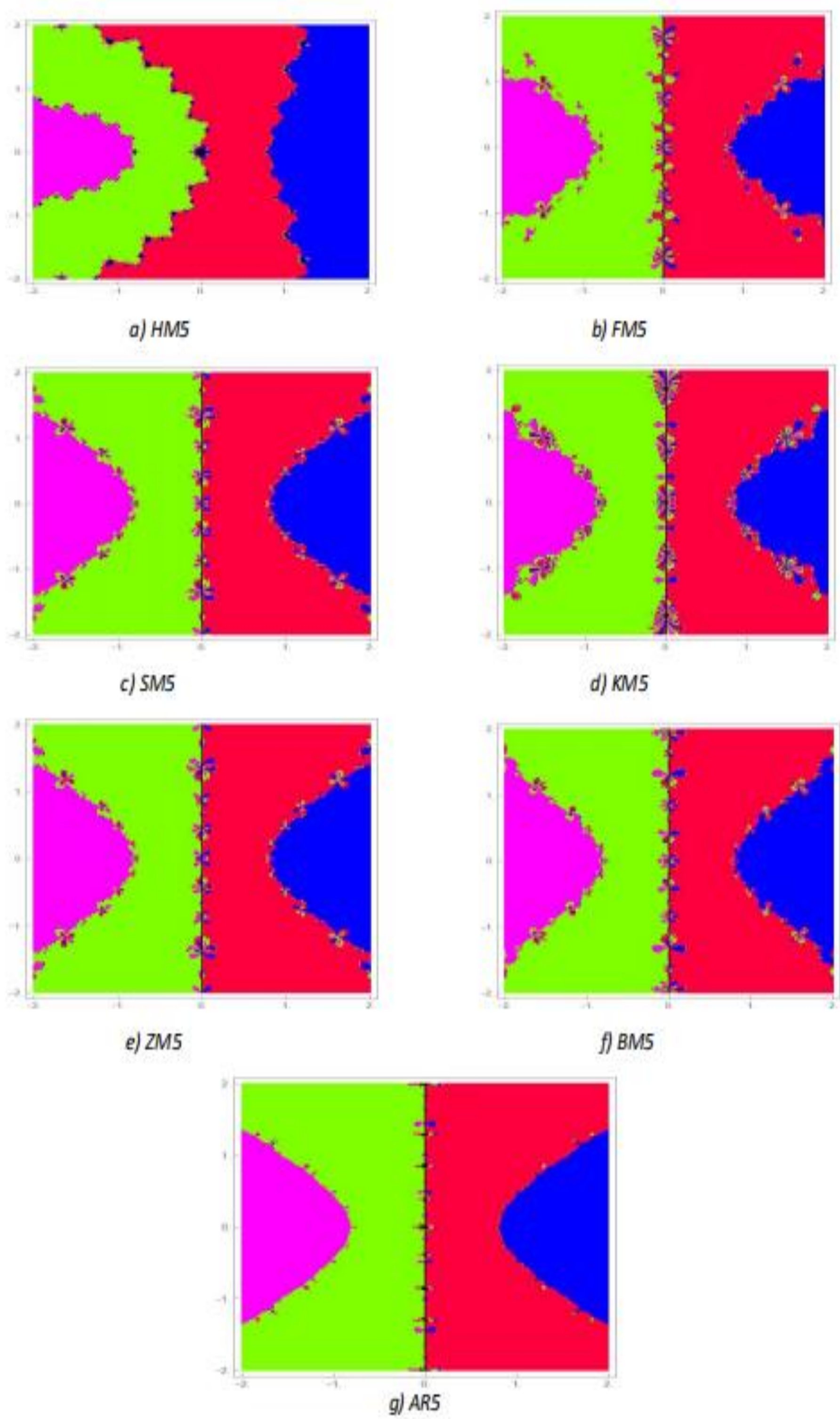



Fig 3 : Basins of attraction for $p_3(z)=z^4-(5/4)z^2+(1/4)$.

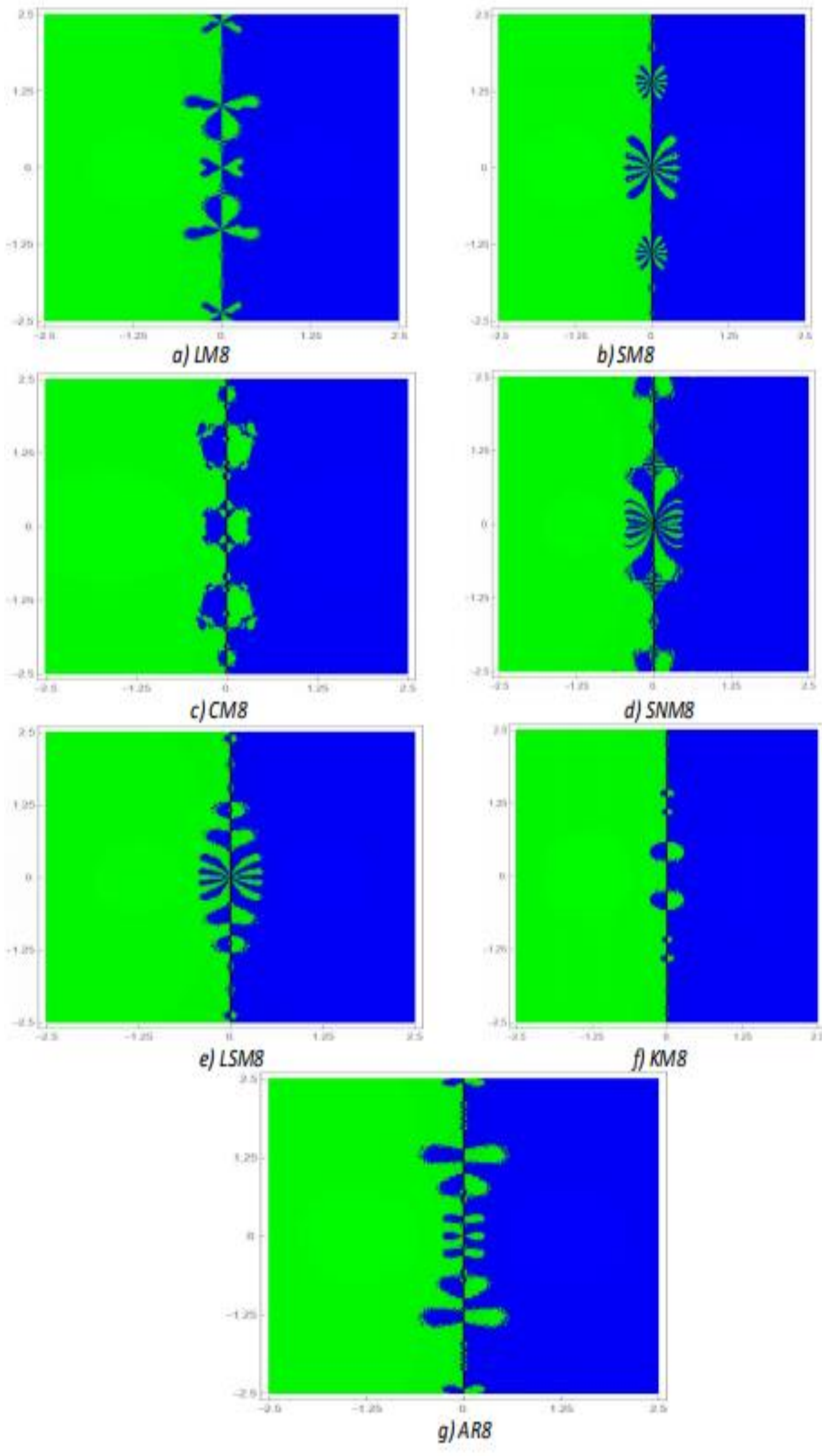
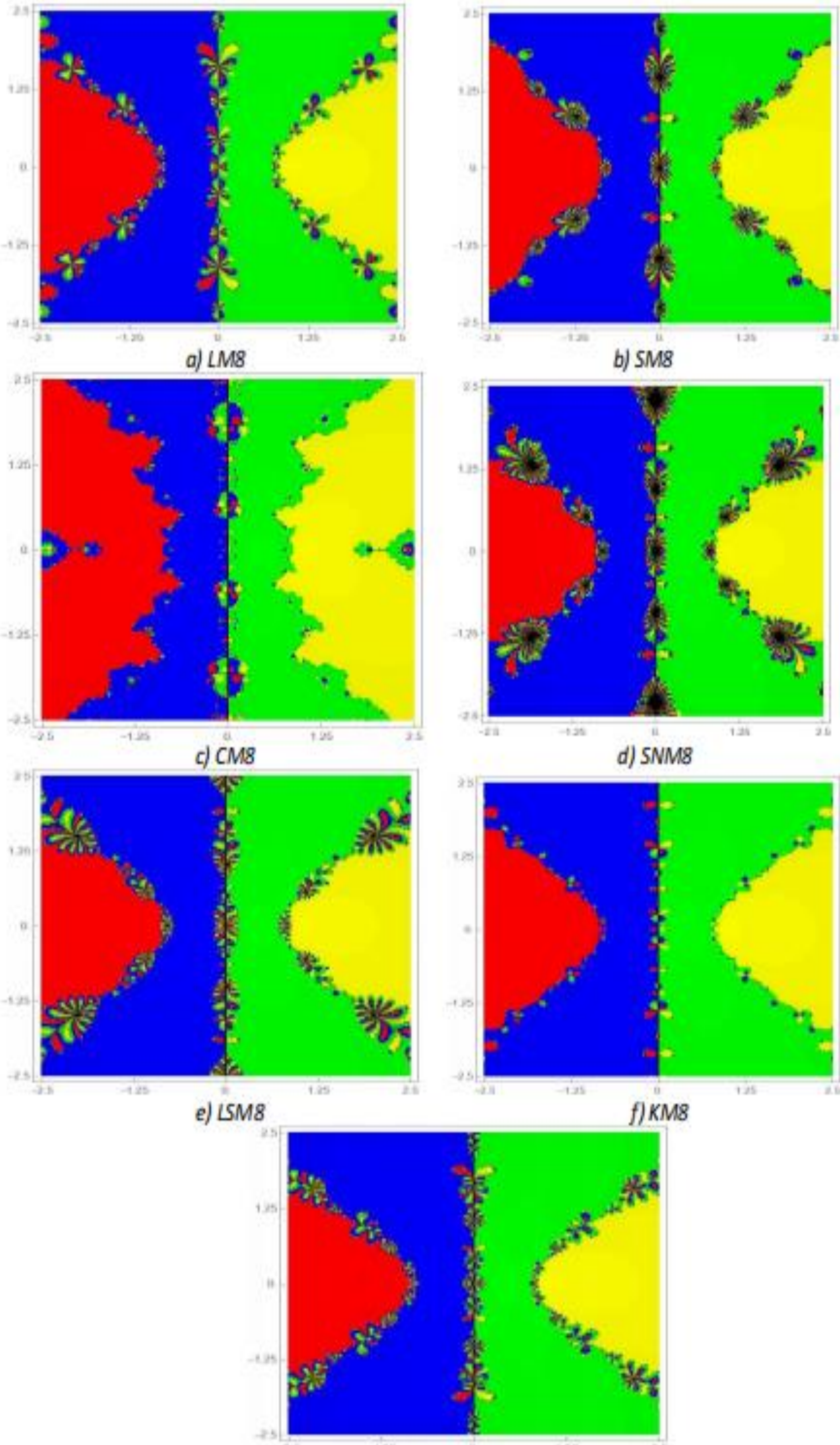



Fig 4 : Basins of attraction for $p_1(z)=z^2-1$.



a) LM8 b) SM8 c) CM8 d) SNM8 e) LSM8 f) KM8 g) AR8

Fig 5: Basins of attraction for $p_3(z)=z^4-(5/4)z^2+(1/4)$ .

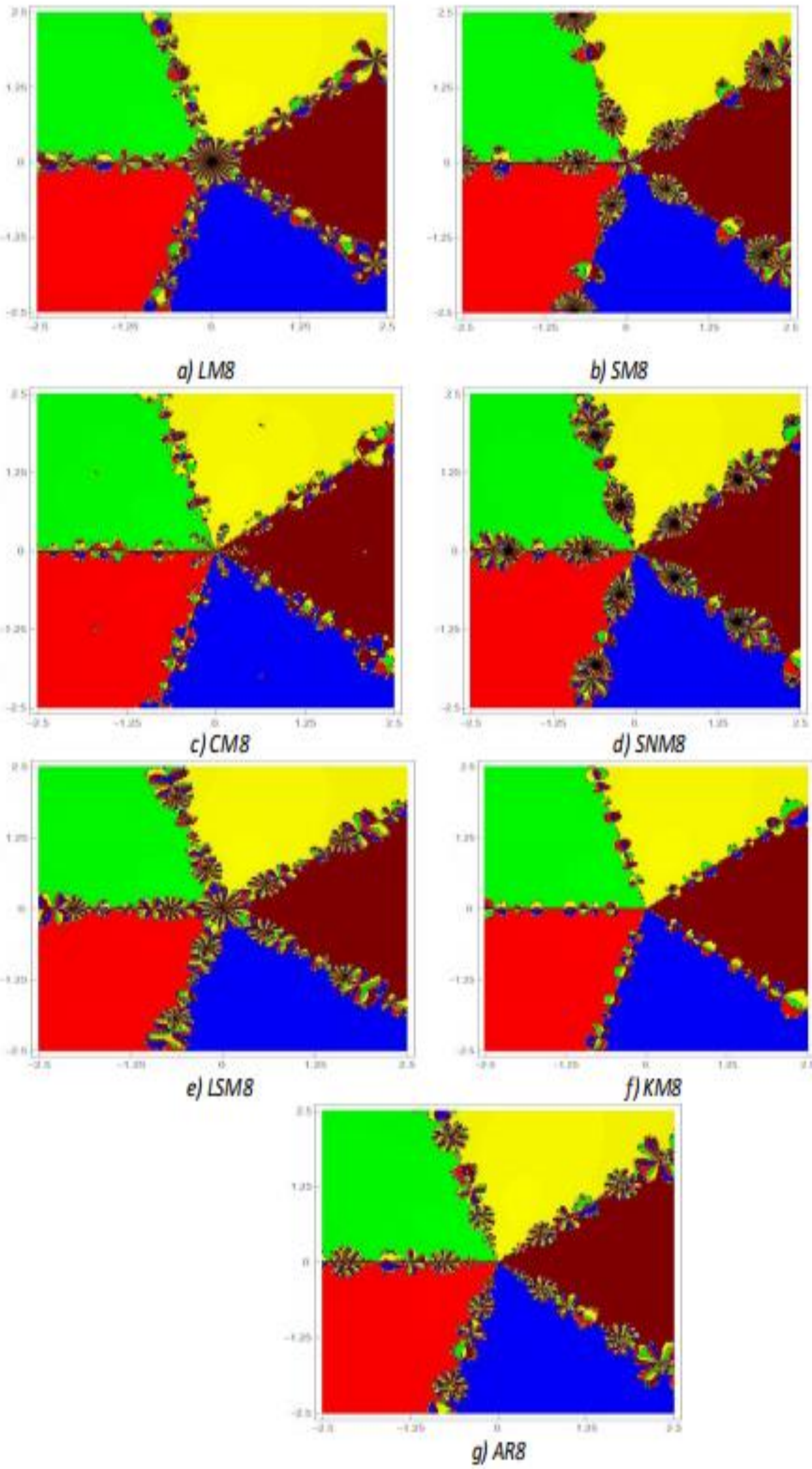



Fig 6: Basins of attraction for $p_4(z)=z^4-1/z$ .

## 6. Conclusion

In this work, we have proposed fifth and eighth order iterative methods for finding simple roots of nonlinear equations. The methods require four and five function evaluations in each iteration to achieve the convergence order five and eight respectively and does not require evaluation of second derivative. Certain numerical experimentation have confirmed the efficiency and robustness of the proposed methods. Moreover, presented basins of attraction reveal the good performance of the proposed methods in comparison to various existing fifth and eighth order methods in literature.

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**Conflicts of Interest:** The authors have no conflicts of interest to declare.

## References

[1] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, NewYork, 1977.

[2] C.T. Kelley, Solving Nonlinear Equations with Newton Method, SIAM, Philadelphia, Pa, USA, 2003.

[3] A.M. Ostrowski, Solution of Equations and Systems of Equations, Net-A-Sof, Monterey,Calif,USA, 1983.

[4] Y. Ham, C. Chun, A fifth-order iterative method for solving nonlinear equations, Appl. Math. Comput. 194(2007) 287–290. doi:10.1016/j.amc.2007.04.005.

[5] L. Fang, L. Sun, G. He, An efficient newton-type method with fifth-order convergence for solving nonlinear equations, J. Comput. Appl. Math. 27 (2008) 269–274.

[6] R. Sharma, Some fifth and sixth order iterative methods for solving nonlinear equations, Int. J. Eng. Res.App. 4 (2014) 268–273.

[7] M. Kumar, A.K. Singh, A. Srivastava, A new fifth order derivative free newton-type method for solvingnonlinear equations, Appl. Math. Inf. Sci. 3 (2015) 1507–1513.

[8] X. Zhang, F. A. Shah, Y. Li, L. Yan, A. Q. Baig, M. R. Farahani, A family of fifth-order convergent methodsfor solving nonlinear equations using variational iteration technique, J. Inf. Opt. Sci. 39 (3) (2018) 673–694.

[9] H. M. Bawazir, Fifth and eleventh-order iterative methods for roots of nonlinear equations, HadhramoutUniversity J.

Nat. Appl. Sci. 17 (2) (2020) 6.

[10] L. Liu, X. Wang, Eighth-order methods with high efficiency index for solving nonlinear equations, Appl.Math. Comput. 215 (9) (2010) 3449–3454.

[11] J. Sharma, H. Arora, An efficient family of weighted-newton methods with optimal eighth order conver- gence, Appl. Math. Letters 29 (2014) 1–6.

[12] A. Cordero, M. Fardi, M. Ghasemi, J.R. Torregrosa, Accelerated iterative methods for finding solutions of nonlinear equations and their dynamical behavior, Calcolo 51 (1) (2014) 17–30.

[13] R. Sharma, J.R. Sharma, N. Kalra, A novel family of weighted-newton optimal eighth order methods withdynamics, SeMA Journal 75 (2018) 197–213.

[14] T. Lotfi, S. Sharifi, M. Salimi, S. Siegmund, A new class of three-point methods with optimal convergenceorder eight and its dynamics, Num. Algo. 68 (2015) 261–288.

[15] B. Kong-ied, Two new eighth and twelfth order iterative methods for solving nonlinear equations, Inter. J.Math. and Comp. Sci. 16 (2021) 333–344.

[16] M. Petković, B. Neta, L. Petković, J. Džunić, Multipoint methods for solving nonlinear equations, Academic press, Amsterdam, The Netherlands, 2012.

[17] B.D. Stewart, Attractor basins of various root-finding methods, Tech. rep., Naval Postgraduate School Mon-terey CA (2001).

[18] J.L. Varona, Graphic and numerical comparison between iterative methods, Math. Intel. 24 (1) (2002) 37–47.