Some Separation Axioms In Nano Ideal Topological Spaces

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ABSTRACT

The notion of nano ideal topological space was introduced by M. Parimala et al [6]. They studied its properties and characterizations. Also they introduced the concept of nano ideal generalized closed sets in nano ideal topological spaces and investigated some of its basic properties. The main goal of this work is to induct and study the properties of $\mathrm{nIgsemi}^*\mathrm{T}_0$-space, $\mathrm{nIgsemi}^*\mathrm{T}_1$-space, $\mathrm{nIgsemi}^*$- connected space and $\mathrm{nIgsemi}^*$-Hausdroff space in nano ideal topological spaces and study its relationship with existing spaces. Further we obtain certain characterizations of these spaces.

Key Words: nano ideal topological spaces,$\mathrm{nIgsemi}^*\mathrm{T}_0$-space, $\mathrm{nIgsemi}^*\mathrm{T}_1$-space, $\mathrm{nIgsemi}^*$-connected space and $\mathrm{nIgsemi}^*$-Hausdroff space.

1. Preliminary

Definition 1.1 [1] A subset A of a nano ideal topological space $(\mathrm{U}, \mathrm{N}, \mathrm{I})$ is said to be **nano ideal generalized semi*-closed** (briefly, $\mathrm{nIgsemi}^*$-closed) if $\mathrm{A}_{\mathrm{n}}^{*} \subseteq \mathrm{V}$ whenever $\mathrm{A} \subseteq \mathrm{V}$ and V is nano semi^*-open. The family of all $\mathrm{nIgsemi}^*$-closed sets of U is denoted by $\mathrm{nIgsemi}^*\mathrm{C}(\mathrm{U}, \mathrm{N}, \mathrm{I})$(or simply $\mathrm{nIgsemi}^*\mathrm{C}(\mathrm{U})$.

Definition 1.2 [1] A subset A of a nano ideal topological space $(\mathrm{U}, \mathrm{N}, \mathrm{I})$ is said to be **nano ideal generalized semi*-open** (briefly, $\mathrm{nIgsemi}^*$-open) if $\mathrm{U} - \mathrm{A}$ is $\mathrm{nIgsemi}^*$-closed. The family of all $\mathrm{nIgsemi}^*$-open sets of U is denoted by $\mathrm{nIgsemi}^*\mathrm{O}(\mathrm{U}, \mathrm{N}, \mathrm{I})$(or simply $\mathrm{nIgsemi}^*\mathrm{O}(\mathrm{U})$).

Definition 1.3 [1] The **nano** I **generalized** $\mathbf{semi}^*$**-closure** of A is defined as the intersection of all nano I generalized semi^*-closed sets containing A and it is denoted by nI generalized semi^*-$\mathrm{cl}(\mathrm{A})$. i.e., $\mathrm{nIgsemi}^*\mathrm{cl}(\mathrm{A}) = \cap \{\mathrm{F} : \mathrm{A} \subseteq \mathrm{F} \in \mathrm{nIgc}(\mathrm{N})$. nI generalized semi^*-$\mathrm{cl}(\mathrm{A})$ is the smallest nano I generalized semi^*-closed set containing$\mathrm{A}$.

Definition 1.4 [3] A map $\mathrm{f}: (\mathrm{U}, \mathrm{N}, \mathrm{I}) \rightarrow (\mathrm{V}, \mathrm{M}, \mathrm{J})$ is said to **be strongly** $\mathbf{nIgsemi}^*$**-open** if the image of every $\mathrm{nIgsemi}^*$-open set in U is $\mathrm{nIgsemi}^*$-open in V.

Definition 1.5 [2] A map $\mathrm{f} : (\mathrm{U}, \mathrm{N}, \mathrm{I}) \rightarrow (\mathrm{V}, \mathrm{M}, \mathrm{J})$ is called $\mathbf{nIgsemi}^*$**-continuous function**, if $\mathrm{f}^{-1}(\mathrm{A})$ of each n-open set $\mathrm{A} \subseteq (\mathrm{V}, \mathrm{M}, \mathrm{J})$ is a $\mathrm{nIgsemi}^*$-open set in $(\mathrm{U}, \mathrm{N}, \mathrm{I})$.

Definition 1.6.[2] Consider two nano ideal topological spaces $(\mathrm{U}, \mathrm{N}, \mathrm{I})$ and $(\mathrm{V}, \mathrm{M}, \mathrm{J})$ and define a function $\mathrm{f}: (\mathrm{U}, \mathrm{N}, \mathrm{I}) \rightarrow (\mathrm{V}, \mathrm{M}, \mathrm{J})$. The function f is defined to be a $\mathbf{nIgsemi}^*$**-irresolute function**, if the inverse image $\mathrm{f}^{-1}(\mathrm{A})$ is a $\mathrm{nIgsemi}^*$-open set in $(\mathrm{U}, \mathrm{N}, \mathrm{I})$ for every $\mathrm{nIgsemi}^*$**-**open set A in $(\mathrm{V}, \mathrm{M}, \mathrm{J})$.

Definition 1.7 [7] A space U is called **nano-**$\mathbf{T_1}$ (or N-T_1) for $\mathrm{x}, \mathrm{y} \in \mathrm{U}$ and $\mathrm{x} \neq \mathrm{y}$, there exists a nano-open sets G and H such that $\mathrm{x} \in \mathrm{G}, \mathrm{y} \notin \mathrm{G}$ and $\mathrm{y} \in \mathrm{H}, \mathrm{x} \notin \mathrm{H}$.

Definition 1.8 [4] A nano topological space $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ is said to be **nano-connected** if $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ cannot be expressed as a disjoint union of two non-empty nano-open sets. A subset of $(\mathrm{U}\tau_{\mathrm{R}}(\mathrm{X}))$ is nano-connected as a subspace. A subset is said to be nano disconnected if and only if it is not nano-connected.

Definition 1.9 [5] A space $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ is said to be **nano-Hausdroff** if whenever x and y are distinct points of $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$, there exists disjoint nano open sets A and B such that $\mathrm{x} \in \mathrm{A}$ and $\mathrm{y} \in \mathrm{B}$.

Definition 1.10 [1] Every nano open set is $\mathrm{nIgsemi}^*$-open.

2. $\mathbf{nIgsemi}^*\mathbf{T_0}$-Spaces

Definition 2.1. A nano ideal topological space U is said to be $\mathrm{nIgsemi}^*\mathrm{T}_0$-space if for each pair of distinct points x and y of U, there exists a $\mathrm{nIgsemi}^*$-open set containing one point but not the other.

Theorem 2.2. A nano ideal topological space U is a $\mathrm{nIgsemi}^*\mathrm{T}_0$-space then $\mathrm{nIgsemi}^*$-closures of distinct points are distinct.

Proof. Let x and y be distinct points of U. Since U is $\mathrm{nIgsemi}^*\mathrm{T}_0$-space, there exists a $\mathrm{nIgsemi}^*$-open set G such that $x \in G$ and $y \notin G$. Consequently, $U - G$ is a $nIgsemi^*$-closed set containing y but not x. But $nIgsemi^*cl\{y\}$ is the intersection of all $nIgsemi^*$- closed sets containing y. Hence $y \in nIgsemi^*cl\{y\}$ but $x \notin nIgsemi^*cl\{y\}$ as $x \notin U - G$. Therefore, $nIgsemi^*cl\{x\} \neq nIgsemi^*cl\{y\}$. Hence $nIgsemi^*$-closures of distinct points are distinct.

Theorem 2.3. If $f: U \rightarrow V$ is a bijection, strongly $nIgsemi^*$-open and U is $nIgsemi^*T_0$-space, then V is also $nIgsemi^*T_0$-space.

Proof. Let y_1 and y_2 be two distinct points of V. Since f is bijective, there exist distinct points x_1 and x_2 of U such that $f(x_1) = y_1$ and $f(x_2) = y_2$ Since U is $nIgsemi^*T_0$-space there exists a $nIgsemi^*$-open set G such that $x_1 \in G$ and $x_2 \notin G$.Therefore $y_1 = f(x_1) \in f(G)$ and $y_2 = f(x_2) \notin f(G)$. Since f is strongly $nIgsemi^*$-open function, $f(G)$ is $nIgsemi^*$-open in V. Thus, there exists a $nIgsemi^*$-open set $f(G)$ in V such that $y_1 \in f(G)$ and $y_2 \notin f(G)$. Therefore V is $nIgsemi^*T_0$-space.

3. $nIgsemi^*T_1$-Spaces

Definition 3.1. A nano topological space U is said to be **$nIgsemi^*T_1$-space** if for any pair of distinct points x and y, there exist a $nIgsemi^*$-open sets G and H such that $x \in G, y \notin G$ and $x \notin H, y \in H$.

Theorem 3.2. In a nano ideal topological space U, if singletons are $nIgsemi^*$- closed sets then U is $nIgsemi^*T_1$-space.

Proof. Assume singletons $\{x\}$ and $\{y\}$ belongs to $nIgsemi^*$-closed sets. Let x and $y \in U$ with $x \neq y$. Now $x \neq y$ implies $y \in U - \{x\}$. Hence $U - \{x\}$ is $nIgsemi^*$- open set containing y but not x. Similarly, we can prove $U - \{y\}$ is $nIgsemi^*$-open set containing x but not y. Therefore U is $nIgsemi^*T_1$-space.

Theorem 3.3. The property being $nIgsemi^*T_1$-space is preserved under bijection and strongly $nIgsemi^*$-open function.

Proof. Let $f: U \rightarrow V$ be bijective and strongly $nIgsemi^*$-open function. Let U be a $nIgsemi^*T_1$-space and y_1, y_2 be any two distinct points of V. Since f is bijective there exist distinct points x_1, x_2 of U such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since U is a $nIgsemi^*T_1$-space, there exist $nIgsemi^*$-open sets G and H such that $x_1 \in G, x_2 \notin G$ and $x_1 \notin H, x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. Now f being strongly $nIgsemi^*$-open, $f(G)$ and $f(H)$ are $nIgsemi^*$-open subsets of V such that $y_1 \in f(G)$ but $y_2 \notin f(G)$ and $y_2 \in f(H)$ and $y_1 \notin f(H)$. Hence V is $nIgsemi^*T_1$-space.

Theorem 3.4. If $f: U \to V$ is $nIgsemi^*$- continuous injection and V is NT_1 then U is $nIgsemi^*T_1$-space.

Proof. Let $f: U \to V$ be $nIgsemi^*$-continuous injection and V be NT_1-space. Since the function f is injection, for any two distinct points x_1, x_2 of U, there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is NT_1 space, there exist nano open sets G and H in V such that $y_1 \in G, y_2 \notin G$ and $y_2 \in H, y_1 \notin H$. That is $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Since f is $nIgsemi^*$-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are $nIgsemi^*$-open sets in U. Thus, for two distinct points x_1, x_2 of U there exist $nIgsemi^*$- open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Therefore U is $nIgsemi^*T_1$-space.

Theorem 3.5. If $f: U \to V$ is $nIgsemi^*$- irresolute injective function and V is $nIgsemi^*T_1$-space then U is $nIgsemi^*T_1$-space.

Proof. Let x_1, x_2 be pair of distinct points in U. Since f is injective there exist distinct points y_1, y_2 of V such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since V is $nIgsemi^*T_1$-space there exist $nIgsemi^*$-open sets G and Y in V such that $y_1 \in G, y_2 \notin G$ and that $y_2 \in H, y_1 \notin H$. That is $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Since f is $nIgsemi^*$-irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are $nIgsemi^*$-open sets in U. Thus, for two distinct points x_1, x_2 of U there exist $nIgsemi^*$- open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_1 \notin f^{-1}(H)$ and $x_2 \in f^{-1}(H)$, $x_2 \notin f^{-1}(G)$. Therefore U is $nIgsemi^*T_1$-space.

4. $nIgsemi^*$- Connected Spaces

Definition 4.1 An nano ideal topological space (U, N, I) is called **$nIgsemi^*$- connected** if (U, N, I) cannot be written as the disjoint union of two non empty $nIgsemi^*$-open subsets of (U, N, I).

Example 4.2 Consider the universal set $U = \{a, b, c, d\}$, the approximation space $U/R = \{\{a\}, \{b\}, \{c, d\}\}, X = \{b, d\} \subseteq U$ with the ideal $I = \{\varphi, \{a\}\}$. The nano topology defined by U is $\tau_R(X) = \{U, \varphi, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $nIgsemi^*$-open sets are $\{\{b\}, \{c\}, \{d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{b, c, d\}, U, \varphi\}$. Here (U, N, I) cannot be expressed as the union of two disjoint non empty $nIgsemi^*$-open subsets of (U, N, I). Thus (U, N, I) is said to be $nIgsemi^*$- connected.

Theorem 4.3. In nano ideal topological space U, $\mathrm{nIgsemi}^*$-continuous image of a $\mathrm{nIgsemi}^*$-connected space is nano connected.

Proof. Assume $f:(U,N,I) \rightarrow (V,M,J)$ be a contra $\mathrm{nIgsemi}^*$-continuous function. Let (U,N,I) be a $\mathrm{nIgsemi}^*$-connected space. We have to prove contra $\mathrm{nIgsemi}^*$-continuous image of a $\mathrm{nIgsemi}^*$-connected space U is nano connected. That is to prove V is nano connected. Suppose, let V be nano disconnected. Considering G and S are nano open sets which form a nano disconnection of V, $V = G \cup S$ where $G \cap S = \phi$. Given that f is $\mathrm{nIgsemi}^*$- continuous, $U = f^{-1}(V)$. Since $V = G \cup S$, $U = f^{-1}(G \cup S) = f^{-1}(G) \cup f^{-1}(S)$. Here both $f^{-1}(G)$ and $f^{-1}(S)$ are non empty $\mathrm{nIgsemi}^*$-open sets in U. Additionally, $f^{-1}(G) \cap f^{-1}(S) = \phi$. Therefore U is not $\mathrm{nIgsemi}^*$-connected. This is a contradiction. Hence V is nano connected.

Lemma 4.4. For an nano ideal topological space (U,N,I), U is $\mathrm{nIgsemi}^*$-connected if and only if the only subset of U which are both $\mathrm{nIgsemi}^*$-open set and $\mathrm{nIgsemi}^*$-closed set are the empty set ϕ and U.

Proof. Let G be a $\mathrm{nIgsemi}^*$-clopen subset of U. Then U - G is $\mathrm{nIgsemi}^*$-clopen set. Assume, U is $\mathrm{nIgsemi}^*$-connected. We have to prove G is either ϕ or U. Suppose $G \neq \emptyset$ and U. Then U can be expressed as disjoint union of two non empty $\mathrm{nIgsemi}^*$-open subsets of both G and U-G, which is contradiction to U is $\mathrm{nIgsemi}^*$-connected. Hence the only subset of U which are both $\mathrm{nIgsemi}^*$-open set and $\mathrm{nIgsemi}^*$-closed set are the empty set and U.

Conversely, Assume the only subset of U which are both $\mathrm{nIgsemi}^*$-open set and $\mathrm{nIgsemi}^*$-closed set are the empty set ϕ and U. We have to prove U is $\mathrm{nIgsemi}^*$-connected. Suppose Assume U is $\mathrm{nIgsemi}^*$-disconnected. Then $U = G \cup S$ where G and S are disjoint non empty $\mathrm{nIgsemi}^*$-open set subsets of U. By our assumption either $G = \phi$ or U which against the assumption G and S are disjoint non empty $\mathrm{nIgsemi}^*$-open subsets of U. Hence U is $\mathrm{nIgsemi}^*$-connected.

5. $\mathrm{nIgsemi}^*$-Hausdroff Spaces

Definition 5.1 A nano ideal topological space (U,N,I) is referred as a **$\mathrm{nIgsemi}^*$-Hausdroff space**, if for any two points $x, y \in U$ and $x \neq y$ there corresponds two $\mathrm{nIgsemi}^*$-open sets K and L such that $K \cap L = \varphi$ and $x \in K, y \in L$.

Theorem 5.2. A nano Hausdroff Space is always a $\mathrm{nIgsemi}^*$-Hausdroff Space.

Proof: The conclusion that comes from the definition of nano Hausdroff space **a**nd by theorem 1.10, A nano open set is always a $\mathrm{nIgsemi}^*$-open set.

Theorem 5.3. Let (U, N, I) be a nano ideal topological space and (V, M, J) be $\mathrm{nIgsemi}^*$-Hausdroff space and if the function $f : (U, N, I) \rightarrow (V, M, J)$ be a $\mathrm{nIgsemi}^*$-irresolute and injective mapping, then U is a $\mathrm{nIgsemi}^*$-Hausdroff space.

Proof. Let $x, y \in U$ and $x \neq y$. Since f is injective, $f(x), f(y)$ are the two distinct images of x and y. Due to the fact that V is a $\mathrm{nIgsemi}^*$-Hausdroff space, there corresponds two disjoint $\mathrm{nIgsemi}^*$-open sets K and L such that $f(x) \in K$ and $f(y) \in L$ and so $x \in f^{-1}(K)$ and $y \in f^{-1}(L)$. Given that f is $\mathrm{nIgsemi}^*$-irresolute and $K \cap L = \varphi$ we obtain that $f^{-1}(K)$ and $f^{-1}(L)$ are the disjoint $\mathrm{nIgsemi}^*$-open sets in U that include the distinct points $x, y \in U$. Hence U is a $\mathrm{nIgsemi}^*$-Hausdroff space.

Theorem 5.4. If the function $f: (U, N, I) \rightarrow (V, M, J)$ is a $\mathrm{nIgsemi}^*$-continuous mapping injective mapping from nano ideal topological space (U, N, I) to nano Hausdroff space (V, M, J), then (U, N, I) is a $\mathrm{nIgsemi}^*$-Hausdroff space.

Proof. Suppose $x, y \in U$ and $x \neq y$. Since f is a one-one mapping, The images of x and y are, respectively, $f(x)$ and $f(y)$, which are also distinct in (V, M, J). Since V is nano-Hausdroff Space, there corresponds nano-open sets K, L and $K \cap L = \varphi$ such that $f(x) \in K$ and $f(y) \in L$. This means that $f^{-1}(K)$ and $f^{-1}(L)$ are disjoint $\mathrm{nIgsemi}^*$-open sets of U as well as x belongs to $f^{-1}(K)$ and y belongs to $f^{-1}(L)$. Hence U is a $\mathrm{nIgsemi}^*$-Hausdroff Space.

Theorem 5.5. If $f: (U, N, I) \rightarrow (V, M, J)$ is $\mathrm{nIgsemi}^*$-continuous and bijective mapping from $\mathrm{nIgsemi}^*$-compact space (U, N, I) to $\mathrm{nIgsemi}^*$-Hausdroff space (V, M, J), then f is $\mathrm{nIgsemi}^*$-Homeomorphic.

Proof. It suffices to prove $f^{-1} = g: V \rightarrow U$ is $\mathrm{nIgsemi}^*$-continuous in order to prove that f is $\mathrm{nIgsemi}^*$-Homeomorphic. To do that, we first prove that pre image $g^{-1}(A)$ is $\mathrm{nIgsemi}^*$-open set ($\mathrm{nIgsemi}^*$-closed) in V for every nano open set A (nano closed set) in U. $U - A$ is nano closed in U because A is nano open. So we have to prove $g^{-1}(U - A)$ $\mathrm{nIgsemi}^*$-closed in V. We may deduce that $g^{-1}(U - A) = V - f(A) \ldots\ldots\ldots\ldots\ldots (1)$, since $g^{-1} = f$.

To prove: $f(A)$ is a $\mathrm{nIgsemi}^*$-open set in V. Since U is $\mathrm{nIgsemi}^*$-compact, a finite subcover corresponds to each $\mathrm{nIgsemi}^*$-open cover. Additionally because V is $\mathrm{nIgsemi}^*$-Hausdroff, two disjoint $\mathrm{nIgsemi}^*$-open sets K and L exists for any two distinct points $x, y \in U$ such that $f(x) \in K$ and $f(y) \in L$,. Each element of U is mapped to distinct elements V since f is bijective. Therefore for every nano open set A in $U, f(A)$ is $nIgsemi^*$-open in V. From (1) for every nano closed set in U, the pre image g is $nIgsemi^*$-closed in V. As a result, $g = f^{-1}$ is a $nIgsemi^*$-continuous mapping. Hence f is $nIgsemi^*$ -Homeomorphic.

Conclusion: In this paper, different types of separation axioms namely $nIgsemi^*T_0$-space, $nIgsemi^*T_1$-space, $nIgsemi^*$- connected space and $nIgsemi^*$-Hausdroff space in nano ideal topological spaces are introduced and studied. Basic properties and characterizations related to these sets are given.

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