Simple Acyclic Graphoidal Covering Number In A Semigraph

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Abstract

A simple graphoidal cover of a semigraph G is a graphoidal cover Ψ of G such that any two paths in Ψ have atmost one end vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of a semigraph and is denoted by $\eta_s(G)$. A simple acyclic graphoidal cover of a semigraph G is an acyclic graphoidal cover Ψ of G such that any two paths in Ψ have atmost one end vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of a semigraph and is denoted by $\eta_{as}(G)$. In this paper we find the simple acyclic graphoidal covering number for wheel in a semigraph, unicycle in a semigraph and zero-divisor graph.

Keywords: graphoidal cover, semigraph, simple acyclic graphoidal cover.

1 Introduction

Sampathkumar and Acharya initially established the idea of graphoidal covers and graphoidal covering number in [1]. Following that, a variety of topics were introduced and thoroughly in- vestigated, including domination in graphoidally covered graphs, acyclic graphoidal covering number, graphoidal graphs, etc.

A semigraph G is a pair (V, X), where V is a non-empty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices, for various $\mathrm{n} \geq 2$, satisfying the following conditions.

S.G.-1 Any two edges have atmost one vertex in common.

S.G.-2 Two edges $(\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_\mathrm{n})$ and $(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\mathrm{m})$ are considered to be equal if and only if

(i) $\mathrm{m} = \mathrm{n}$ and

(ii) either $\mathrm{u}_\mathrm{i} = \mathrm{v}_\mathrm{i}$, for $1 \leq \mathrm{i} \leq \mathrm{n}$, or $\mathrm{u}_\mathrm{i} = \mathrm{v}_{\mathrm{n}-\mathrm{i}+1}$, for $1 \leq \mathrm{i} \leq \mathrm{n}$. [6]

Let $\mathrm{G} = (\mathrm{V}, \mathrm{X})$ be a semigraph and $\mathrm{E} = (\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\mathrm{n})$ be an edge of G. Then v_1 and v_n are the end vertices of E and v_i, $2 \leq \mathrm{i} \leq \mathrm{n} - 1$ are the middle vertices (or m-vertices) of E. Further, we say that if a vertex v of a semigraph G appears only as an end vertex then it is an end vertex. If a vertex v is only a middle vertex then it is a middle vertex while a vertex v is called middle-cum-end ((m, e)-vertex) if it is a middle vertex of some edge and end vertex ofsome edge. [6]

If a vertex v is an m-vertex of more than one edge of G, say $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_\mathrm{n}$, then v is represented as a small regular polygon with 2t corners $\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_\mathrm{n}$ with the convention that the Jordan curverepresenting the edge E_i, meets the polygon precisely at c_i and $\mathrm{c}_{\mathrm{t}+\mathrm{i}}$, $\mathrm{i} + \mathrm{t}$ reduced modulo 2t, $\mathrm{i} \in \{1, 2, \ldots, \mathrm{t}\}$ [6]. A vertex v in a semigraph G is a pendant vertex if $\deg \mathrm{v} = \deg_\mathrm{e} \mathrm{v} = 1$. A pendant edge E is one having a pendant vertex. Thus, a pendant edge has atleast one end vertex which is pendant vertex [6]. A dendroid is a connected semigraph without strong cycles [6]. $\deg \mathrm{v}$ is the number of edges having v as an end vertex and $\deg_\mathrm{e} \mathrm{v}$ is the number of edges containing v [6].

Let $\mathrm{P} = (\mathrm{v}_1, \mathrm{u}_1, \mathrm{u}_2, \mathrm{v}_2, \mathrm{u}_3, \mathrm{u}_4, \mathrm{u}_5, \ldots, \mathrm{u}_{\mathrm{n}-1}, \mathrm{u}_\mathrm{n}, \mathrm{v}_\mathrm{n})$ be a path with $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\mathrm{n}$ as end vertices and $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_\mathrm{n}$ as middle vertices. For convenience let us denote this path by $\mathrm{P} = (\mathrm{v}_1, \mathrm{E}_1, \mathrm{v}_2, \mathrm{E}_2, \ldots, \mathrm{E}_\mathrm{n}, \mathrm{v}_\mathrm{n})$, where $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_\mathrm{n}$ denote the middle vertices between two end vertices. If $\mathrm{P} = (\mathrm{v}_0, \mathrm{E}_1, \mathrm{v}_1, \ldots, \mathrm{E}_\mathrm{n}, \mathrm{v}_\mathrm{n})$ and $\mathrm{Q} = (\mathrm{v}_\mathrm{n} = \mathrm{w}_0, \mathrm{F}_1, \mathrm{w}_1, \mathrm{F}_2, \ldots, \mathrm{F}_\mathrm{m}, \mathrm{w}_\mathrm{m})$ are two paths in G, then the walk obtained by concatating P and Q at v_n is denoted by $\mathrm{P} \circ \mathrm{Q}$ and the path $\mathrm{v}_\mathrm{n}, \mathrm{E}_\mathrm{n}, \ldots, \mathrm{v}_1, \mathrm{E}_1, \mathrm{v}_0$ is denoted by P^{-1} .

Let G be a unicyclic semigraph attached by a dendroid with n pendant vertices. Let C be the unique cycle in G. Let w be a vertex of degree $\mathrm{deg_e}$ greater than 2 on C. $\mathrm{T_i}$, $1 \leq \mathrm{i} \leq \mathrm{k}$ are called the branches of G, where k denote the number of pendant vertices in the dendroid attached to w.

Definition 1.1. A Wheel in a semigraph is formed by connecting an end vertex to all the end vertices of a cycle. The Wheel in a semigraph is denoted by $\mathrm{W_n}$, where n is the number of end vertices. An end vertex connecting the end vertex of a cycle may or may not contain $\mathrm{m}-$vertices.

Remark 1.2. [9] Obviously for any tree T, we have $\eta = \eta_a = \eta_s = \eta_{as} = \mathrm{n} - 1$, where n is the number of pendant vertices of T.

Theorem 1.3. [9] Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m denote the number of vertices of degree greater than 2 on C. Then

$$\eta_{as}(\mathrm{G}) = \begin{cases} 3 & \text{if } \mathrm{m} = 0 \\ \mathrm{n}+2 & \text{if } \mathrm{m} = 1 \\ \mathrm{n}+1 & \text{if } \mathrm{m} = 2 \\ \mathrm{n} & \text{if } \mathrm{m} \geq 3 \end{cases}$$

2. Simple Acyclic Graphoidal Covers in Semigraph

Definition 2.1. A graphoidal cover of a semigraph G is a collection Ψ of non-trivial paths (which are not necessarily open) in G satisfying the following conditions:

(i) Every path in Ψ has atleast two end vertices.

(ii) Every end vertex of G is an internal vertex of atmost one path in Ψ.

(iii) Every edge of G is in exactly one path in Ψ.

The set of all graphoidal covers of a semigraph $\mathrm{G} = (\mathrm{V}, \mathrm{X})$ is denoted by $\mathcal{G}_\mathrm{G}$. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of a semigraph and is denoted by $\eta(\mathrm{G})$.

Definition 2.2. An acyclic graphoidal cover of a semigraph G is a graphoidal cover Ψ of G such that every element of Ψ is a path in G. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of a semigraph and is denoted by $\eta_a(\mathrm{G})$.

Definition 2.3. A simple graphoidal cover of a semigraph G is a graphoidal cover Ψ of G such that any two paths in Ψ have atmost one end vertex in common. The minimum cardinality of a simple graphoidal cover of G is called the simple graphoidal covering number of a semigraph and is denoted by $\eta_s(G)$.

Definition 2.4. A simple acyclic graphoidal cover of a semigraph G is an acyclic graphoidal cover Ψ of G such that any two paths in Ψ have atmost one end vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of a semigraph and is denoted by $\eta_{as}(G)$.

Definition 2.5. An end vertex(or middle vertex or middle-cum-end vertex) of G is said to be in the interior of Ψ if it is an internal vertex of some path in Ψ. Any end vertex which is not in the interior of Ψ is said to be in the exterior of Ψ.

Note 2.6. For any path P in a semigraph, let $t(P)$ denote the number of internal vertices which are end vertices of P, so that $t(P) = |E(P)| - 1$.

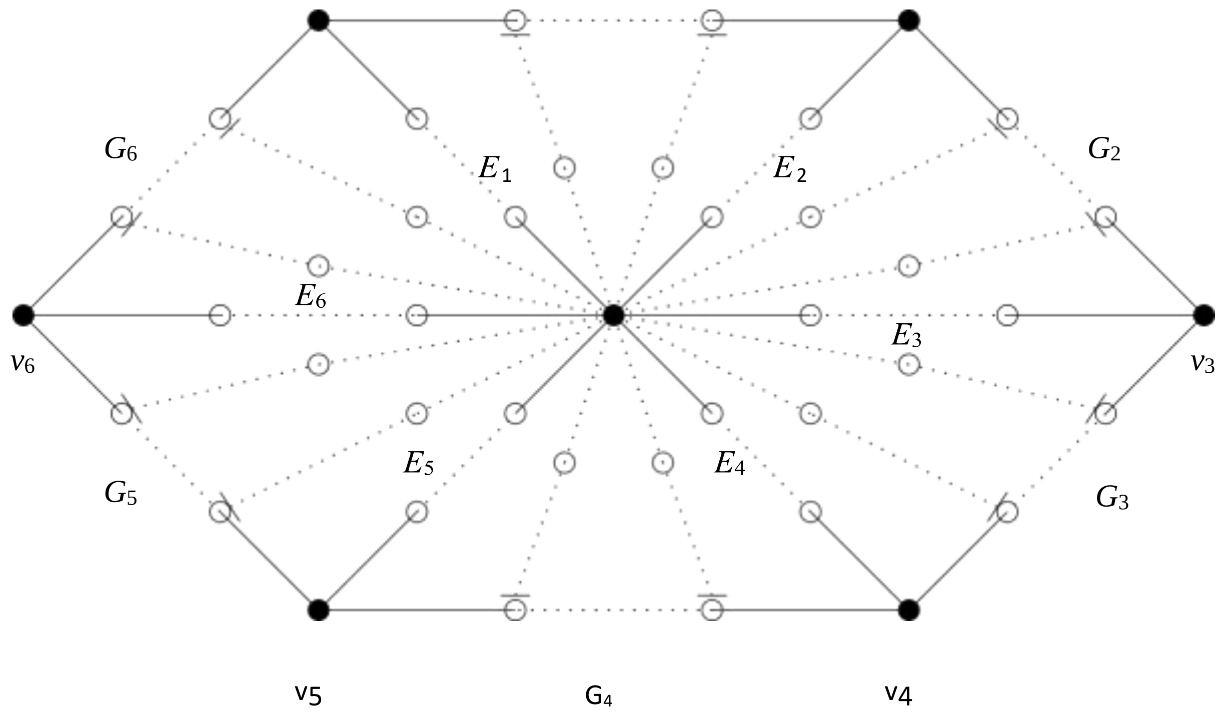
Theorem 2.7. For any simple acyclic graphoidal cover Ψ of a semigraph G, let t_Ψ denote the number of exterior vertices which are end vertices of Ψ. Let t = mint_Ψ, where the minimum is taken over all simple acyclic graphoidal covers Ψ of G. Then $\eta_{as} = q - p + t$, where p is the number of end vertices of G.

Theorem 2.8. Let W_n be the wheel in a semigraph with n end vertices such that every middle vertex in the cycle is attached by the end vertex, then

$$\eta_{as}(W_n) = \begin{cases} m + 6 & if\ n = 4 \\ n + m + 1 & if\ n \geq 5 \end{cases}$$

where m is the number of middle vertices in the cycle.

v1 G_1 v2



Proof. Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the end vertices of W_n and $u_1, u_2, \ldots, u_m$ be the middle vertices in the cycle of W_n and $X(W_n) = \{(v_0, E_i, v_i); 1 \leq i \leq n-1\} \cup \{(v_0, F_i, u_i); 1 \leq i \leq m\} \cup \{(v_i, G_i, v_{i+1}); 1 \leq i \leq n-2\} \cup \{(v_1, G_{n-1}, v_{n-1})\}$.

If $n = 4$, then $\Psi = \{(v_0, F_i, u_i); 1 \leq i \leq m\} \cup \{(v_0, E_i, v_i); 1 \leq i \leq 3\} \cup \{(v_1, G_1, v_2), (v_2, G_2, v_3), (v_3, G_3, v_1)\}$. Therefore $|\Psi| = m + 6$.

Now, suppose $n \geq 5$. Let $P_1 = (v_1, G_1, v_2, G_2, v_3, \ldots, G_{n-3}, v_{n-2})$, $P_2 = \{(v_{n-2}, G_{n-2}, v_{n-1}), (v_{n-1}, G_{n-1}, v_1)\}$, $P_3 = \{(v_0, F_i, u_i); 1 \leq i \leq m\}$, $P_4 = \{(v_0, E_i, v_i); 1 \leq i \leq n-1\} - \{(v_{n-3}, E_{n-3}, v_0), (v_0, E_{n-1}, v_{n-1})\}$ and $P_5 = (v_{n-3}, E_{n-3}, v_0, E_{n-1}, v_{n-1})$. Therefore $\Psi = \{P_1, P_2, P_3, P_4, P_5\}$. This implies that $|\Psi| = 2 + m + n - 3 + 2 = n + m + 1$. Hence, $\eta_{as} \leq n + m + 1$.

Further, for any simple acyclic graphoidal cover Ψ of W_n, atleast three end vertices on C are exterior to Ψ so that $t \geq 3$. Hence $\eta_{as}(W_n) \geq q - p + 3 = n - 1 + m + n - 1 - n + 3$. Thus, $\eta_{as}(W_n) = n + m + 1$. ■

Theorem 2.9. Let G be a unicyclic semigraph attached by a dendroid with n pendant vertices. Let C be the unique cycle in G and let x denote the number of end vertices of degree deg_e greater than 2 on C and let y denote the number of m-vertices of degree deg_e greater than 1 on C. Then

$$\eta_{as}(G) = \begin{cases} 3 & if\ x = 0\ and\ y = 0 \\ n+3 & if\ x = 0\ and\ y \geq 1 \\ n+2 & if\ x = 1\ and\ y \geq 0 \\ n+1 & if\ x = 2\ and\ y \geq 0 \\ n & if\ x \geq 3\ and\ y \geq 0 \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_k$ and $u_1, u_2, \dots, u_m$ be the end vertices and m-vertices of C respectively. Let $E_1, E_2, \dots, E_k$ denote the edges with m-vertices between $(v_1, v_2), (v_2, v_3), \dots, (v_k, v_1)$ respectively.

Case(1). $x = 0$ and $y = 0$. Then $G = C$ and $\eta_{as}(G) = 3$.

Case(2). $x = 0$ and $y \geq 1$.

Let $u_1, u_2, \dots, u_j$, $1 \leq j \leq m$ be the m-vertices on C attached by the dendroid with $m_1, m_2, \dots, m_j$ pendant vertices respectively so that $u_1, u_2, \dots, u_j$ becomes the (m, e)-vertex. Therefore $m_1 + m_2 + \dots + m_j = n$.

Let T'_{1_i} , $1 \leq i \leq m_1$, T'_{2_i} , $1 \leq i \leq m_2$,..., T'_{j_i} , $1 \leq i \leq m_j$ be the branches of G at $u_1, u_2, \dots, u_j$ respectively. Let $\Psi'_{1_i}, \Psi'_{2_i}, \dots, \Psi'_{j_i}$, $1 \leq j \leq m$ be a minimum simple acyclic graphoidal cover of the branches $T'_{1_i}, T'_{2_i}, \dots, T'_{j_i}$, $1 \leq j \leq m$ respectively.

Let $Q_1 = (v_1, E_1, v_2, \dots, E_{k-2}, v_{k-1})$

$Q_2 = (v_{k-1}, E_{k-1}, v_k)$

$Q_3 = (v_k, E_k, v_1)$

Then $\Psi = \left\{\bigcup_{i=1}^{m_1}(\Psi'_{1_i}) \cup \bigcup_{i=1}^{m_2}(\Psi'_{2_i}) \cup \dots \cup \bigcup_{i=1}^{m_j}(\Psi'_{j_i})\right\} \cup \{Q_1, Q_2, Q_3\}$ is a simple acyclic graphoidal cover Ψ of G. Hence $\eta_{as} \leq n + 3$.

Further for any simple acyclic graphoidal cover Ψ of G, the n pendant vertices and at least three end vertices on C are exterior to Ψ, so that $t \geq n + 3$. Hence $\eta_{as} \geq n$ +3. Thus $\eta_{as} = n + 3$.

Case(3). $x = 1$ and $y \geq 0$.

Let v_1 be the unique end vertex of degree dege greater than 2 on C and let v_1 be attached by a dendroid with t pendant vertices.

Let T_i, $1 \leq i \leq k$ be the branches of G at v_1. Let Ψ_i, $1 \leq i \leq k$ be a minimum simple acyclic graphoidal cover of the branch T_i . Let P_1 be the path in Ψ_1 having the end vertex v_1 as a terminal vertex.

Also, the m-vertices on C may or may not be attached by a dendroid. If the m-vertices on C is attached by a dendroid, then the construction is made similar as discussed in case(2). Therefore $t + m_1 + m_2 + \ldots + m_j = n$.

Let $Q_1 = P_1 \circ (v_1, E_1, v_2)$

$Q_2 = (v_2, E_2, v_3, \ldots, E_{k-1}, v_k)$

$Q_3 = (v_k, E_k, v_1)$

Then $\Psi = \left(\bigcup_{i=1}^{t}(\Psi_i) - \{P_1\}\right)\left\{\bigcup_{i=1}^{m_1}(\Psi'_{1_i}) \cup \bigcup_{i=1}^{m_2}(\Psi'_{2_i}) \cup \ldots \cup \bigcup_{i=1}^{m_j}(\Psi'_{j_i})\right\} \cup \{Q_1, Q_2, Q_3\}$ is a simple acyclic graphoidal cover Ψ of G. Hence $\eta_{as} \leq n + 2$.

Further for any simple acyclic graphoidal cover Ψ of G, the n pendant vertices and at least two end vertices on C are exterior to Ψ, so that $t \geq n + 2$. Hence $\eta_{as} \geq n + 2$. Thus $\eta_{as} = n + 2$.

Case(4). $x = 2$ and $y \geq 0$.

Let v_1 and v_r, $1 < r \leq k$ be the end vertices of degree deg_e greater than 2 on C.

Let S_1 and S_2 denote respectively the (v_1, v_r)-section and (v_r, v_1)-section of the cycle C and let v_s be an internal vertex which is the end vertex of S_1(say). Let R_1 and R_2 denote the
(v_1, v_s)-section of S_1 and (v_s, v_r)-section of S_1 respectively.

Let v_1 be attached by a dendroid with b pendant vertices and v_r be attached by a dendroid with c pendant vertices.

Let Ψ_i and Ψ'_j, where $1 \leq i \leq b$ and $1 \leq i \leq c$ be the minimum simple acyclic graphoidal covers of the branches T_i and T'_j of G at v_1 and v_r respectively. Let P_1 and P'_1 denote respectively the paths in Ψ_1 and Ψ'_1 having the end vertices v_1 and v_r as terminal vertices.

Also, the m-vertices on C may or may not be attached by a dendroid. If the m-vertices on C is attached by a dendroid, then the construction is made similar as discussed in case(2). Therefore $b + c + m_1 + m_2 + \ldots + m_j = n$.

Let $Q_1 = P_1 \circ R_1$

$Q_2 = P'_1 \circ R_2^{-1}$

$Q_3 = S_2$

Then $\Psi = \left(\bigcup_{i=1}^{b}(\Psi_i) - \{P_1\}\right) \cup \left(\bigcup_{i=1}^{c}(\Psi'_j) - \{P'_1\}\right) \cup \left\{\bigcup_{i=1}^{m_1}(\Psi'_{1_i}) \cup \bigcup_{i=1}^{m_2}(\Psi'_{2_i}) \cup \ldots \cup \bigcup_{i=1}^{m_j}(\Psi'_{j_i})\right\} \cup \{Q_1, Q_2, Q_3\}$ is a simple acyclic graphoidal cover Ψ of G. Hence $\eta_{as} \leq n + 1$.

Further for any simple acyclic graphoidal cover Ψ of G, the n pendant vertices and at least one end vertex on C are exterior to Ψ, so that $t \geq n + 1$. Hence$\eta_{as} \geq n + 1$.

Thus $\eta_{as} = n + 1$.

Case(5). $x \geq 3$ and $y \geq 0$.

Let $v_1, v_2, \ldots, v_r$, $1 \leq r \leq k$ be the end vertices of degree deg_e greater than 2 on C. Let $v_1, v_2, \ldots, v_r$ be attached by a dendroid with pendant vertices $n_1, n_2, \ldots, n_r$ respectively.

Let $\Psi_{1_i}, \Psi_{2_i}, \ldots, \Psi_{r_i}$, $1 \leq r \leq k$ be the minimum simple acyclic graphoidal covers of the branches T_{1_i}, $1 \leq i \leq n_1$, T_{2_i}, $1 \leq i \leq n_2$, ..., T_{r_i}, $1 \leq i \leq n_r$ of G at $v_1, v_2, \ldots, v_r$ respectively.

Let P_1, P_2 and P_3 respectively denote the paths in Ψ_{1_1}, Ψ_{2_1}, Ψ_{3_1} having v_1, v_2 and v_3 as terminal vertices.

Also, the m-vertices on C may or may not be attached by a dendroid. If the m-vertices on C is attached by a dendroid, then the construction is made similar as discussed in case(2). Therefore $n_1 + n_2, + \ldots + n_r + m_1 + m_2 + \ldots + m_j = n$.

Let $Q_1 = P_1 \circ (v_1, E_1, v_2)$

$Q_2 = P_2 \circ (v_2, E_2, v_3)$

$Q_3 = P_3 \circ (v_3, E_3, \ldots, E_k, v_1)$

Then $\Psi = \left(\bigcup_{i=1}^{n_1}(\Psi_{1_i}) - \{P_1\}\right) \cup \left(\bigcup_{i=1}^{n_2}(\Psi_{2_i}) - \{P_2\}\right) \cup \left(\bigcup_{i=1}^{n_3}(\Psi_{3_i}) - \{P_3\}\right) \cup \left(\bigcup_{i=1}^{n_4}(\Psi_{4_i})\right) \cup \ldots \cup \left(\bigcup_{i=1}^{n_r}(\Psi_{r_i})\right) \cup \left\{\bigcup_{i=1}^{m_1}(\Psi'_{1_i}) \cup \bigcup_{i=1}^{m_2}(\Psi'_{2_i}) \cup \ldots \cup \bigcup_{i=1}^{m_j}(\Psi'_{j_i})\right\} \cup \{Q_1, Q_2, Q_3\}$ is a simple acyclic graphoidal cover Ψ of G such that every vertex of degree deg_e greater than 1 is interior to Ψ and hence $\eta_{as}(G) = n$. ■

Theorem 2.10. A semigraph G has a simple acyclic graphoidal cover satisfying the helly property if G contains no cycle with three end vertices.

Proof. Let G contains cycle with three end vertices, say $C = (x, y, z, t)$. Let Ψ be any simple acyclic graphoidal cover of . Then the edges xEy, yFz and zGt lie on three different paths in Ψ, say P_1, P_2 and P_3 respectively.

Clearly, $\{P_1, P_2, P_3\}$ is a pairwise intersecting family of paths in Ψ. If there exists an end vertex u which is common to the paths P_1, P_2 and P_3 then the end vertices u and y are common to both P_1 and P_2, which is a contradiction. Hence $V(P_1) \cap V(P_2) \cap V(P_3) = \emptyset$. Thus, Ψ does not satisfy the helly property.

■

2 Simple Acyclic Graphoidal Cover in a Zero-Divisor Graph

Definition 3.1.[9] A simple acyclic graphoidal cover of a graph G is an acyclic graphoidal cover Ψ of G such that any two paths in Ψ have atmost one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of a graph and is denoted by $\eta_{as}(G)$ or simply η_{as}.

Definition 3.2. Let R be a commutative ring(with 1) and let $Z(R)$ be its set of zero-divisors. An element $a \in R$ is called a zero-divisor if there exists a non-zero element $b \in R$ such that $a.b = 0$. Let R be a commutative ring with non-zero identity and let $Z(R)$ be its sets of zero-divisors. The zero-divisor graph of R denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) - 0$, the non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$.

Theorem 3.3. For any star $\Gamma(Z_{2p})$ with p vertices, then $\eta_{as}(\Gamma(Z_{2p})) = p - 2$. Further, for the G_a graph of $\Gamma(Z_{2p})$, $\eta_{as}(\Gamma(Z_{2p})) = p - 1$, where p is a prime number and $p > 3$.

Proof. Given that $\Gamma(Z_{2p})$ is a star with p verices. We know that $\Gamma(Z_{2p})$ is a tree with $p - 1$ pendant vertices. By remark 1.2, we have $\eta_{as}(\Gamma(Z_{2p})) = p - 2$. Using the simple acyclic graphoidal cover, we cam construct the G_a graph for the corresponding $\Gamma(Z_{2p})$. The G_a graph for $\Gamma(Z_{2p})$) is a cycle C_3 attached by $p - 3$ pendant vertices to a verrtex in C_3. By case 2 of theorem 1.3, we have for the G_a graph of $\eta_{as}\left(\Gamma\left(Z_{2p}\right)\right) = p - 3 + 2 = p - 1$. ■

Theorem 3.4. $\eta_{as}(\Gamma(Z_9) \times P_n) = 2n - 2$, for all $n \geq 3$.

Proof. We know that $\Gamma(Z_9)$ is isomorphic to P_2.

Let $P_2 = (x_1, x_2)$

$P_n = (y_1, y_2, \ldots, y_n)$

When $n = 3$, let $u_{ij} = (x_i, y_j)$, $1 \leq i \leq 2$; $1 \leq j \leq 3$.

Then

$\{(u_{11}, u_{21}, u_{22}, u_{23}, u_{13}), (u_{11}, u_{12}), (u_{12}, u_{13}), (u_{12}, u_{22})\}$ is a minimum simple acyclic graphoidal cover of $\Gamma(Z_9) \times P_n$.

Let $n \geq 3$ and $u_{ij} = \left(x_i, y_j\right)$, $1 \leq i \leq 2$; $1 \leq j \leq n$, then $\{(u_{11}, u_{21}, u_{22}, \ldots, u_{2n}, u_{1n}), (u_{11}, u_{12}), (u_{12}, u_{13}), \ldots, (u_{1n-1}, u_{2n}), (u_{12}, u_{22}), (u_{13}, u_{23}), \ldots, (u_{1n-1}, u_{2n-1})\}$ is a set of internally disjoint paths without exterior vertices. Thus, $\eta as(\Gamma(Z_9) \times P_n) = 2n - 2$. ■

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