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New Separation Axioms In Intuitionistic Topological Spaces

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Abstract

The idea of this paper is to introduce the notion of separation axioms of intuitionistic i-open sets in two different ways and discuss their properties. Further we illustrate the relationship within T_{i1} and T_{i2} spaces with some examples.

Keywords : $\mathcal{J}$i-T_0 space, $\mathcal{J}$i-T_1 space, $\mathcal{J}$i-T_2 space, T_{i0}, T_{i1} , T_{i2}

1. Introduction

Coker introduced the concept of intuitionistic sets, intuitionistic points[1] and intuitionistic topological spaces[2]. Later he defined T_1 and T_2 separation axioms[3] and discussed some properties. Tamanna Tasnim Prova and Md. Sahadat Hossain[6] defined T_0, T_1 and T_2 spaces as in general topology. Suganya etal[5] defined $\mathcal{J}$i-open set in intutionistic topological spaces and explained some properties. The aim of this paper is to defined another type of Separation axioms depends on $\mathcal{J}$i-open set in intutionistic topological spaces. Here, we discuss T_{i0}, T_{i1} and T_{i2} spaces and discuss the relationship between them. Further, some characterizations of these spaces are also given. Also, we gave many counter examples to prove the reverse implication is not true.

2. Preliminaries

Throughout the present study, $\mathcal{J}$ means intuitionistic, a space A means intuitionistic topological space (A, τ_{I_1}) or (A, τ_I) and B means an intuitionistic topological space (B, τ_{I_2}) unless otherwise mentioned.

Definition 2.1.[3] An Intuitionistic Topological Space (A, τ_I) is said to be

(a) $T_1(i)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{a} \in G$ and $\tilde{b} \notin G$ and $\tilde{b} \in H$ and $\tilde{a} \notin H$.

(b) $T_1(ii)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{a}} \in G$ and $\tilde{\tilde{b}} \notin G$ and $\tilde{\tilde{b}} \in H$ and $\tilde{\tilde{a}} \notin H$.

(c) $T_1(iii)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{a} \in G \subseteq \bar{\tilde{b}}$ and $\tilde{b} \in H \subseteq \bar{\tilde{a}}$.

(d) $T_1(iv)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{a}} \in G \subseteq \bar{\tilde{\tilde{b}}}$ and $\tilde{\tilde{b}} \in H \subseteq \bar{\tilde{\tilde{a}}}$.

(e) $T_1(v)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{b} \notin G$ and $\tilde{a} \notin H$.

(f) $T_1(vi)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{b}} \notin G$ and $\tilde{\tilde{a}} \notin H$.

Definition 2.2.[3] An Intuitionistic Topological Space (A, τ_I) is said to be

(a) $T_2(i)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{a} \in G$, $\tilde{b} \in H$ and $G \cap H = \emptyset$.

(b) $T_2(ii)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{a}} \in G$, $\tilde{\tilde{b}} \in H$ and $G \cap H = \emptyset$.

(c) $T_2(iii)$-space if for all $a, b \in A(a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{a} \in G$, $\tilde{b} \in H$ and $G \subseteq \bar{H}$.

(d) $T_2(iv)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{a}} \in G$, $\tilde{\tilde{b}} \in H$ and $G \subseteq \bar{H}$.

(e) $T_2(v)$-space if for all $a, b \in A(a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{a} \in G \subseteq \bar{\tilde{b}}$, $\tilde{b} \in H \subseteq \bar{\tilde{a}}$ and $G \subseteq \bar{H}$.

(f) $T_2(vi)$-space if for all $a, b \in A\ (a \neq b)$ there exist an intuitionistic open set G, H such that $\tilde{\tilde{a}} \in G \subseteq \bar{\tilde{\tilde{b}}}$, $\tilde{\tilde{b}} \in H \subseteq \bar{\tilde{\tilde{a}}}$ and $G \subseteq \bar{H}$.

Definition 2.3.[6] An intuitionistic topological space (A, τ_I) is called intuitionistic T_0 space
if for all $a, b \in A$ with $a \neq b$ there exist an $\mathcal{J}i$-open set G such that $a \in G_1, b \in G_2$ or $b \in G_1, a \in G_2$.

Definition 2.4.[6] An intuitionistic topological space (A, τ_I) is called intuitionistic T_1 space if for all $a, b \in A$ with $a \neq b$ there exists an $\mathcal{J}$i-open set G, H such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$.

Definition 2.5.[6] An intuitionistic topological space (A, τ_I) is called intuitionistic T_2 space if for all $a, b \in A$ with $a \neq b$ there exists an $\mathcal{J}$i-open set G, H such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = \widetilde{\emptyset}$.

Definition 2.6.[5] An intuitionistic set D of an Intuitionistic topological space (A, τ_I) is said to be an intuitionistic i-open set (shortly $\mathcal{J}$i-open set) if there exist an intuitionistic open set $H \neq \widetilde{\emptyset}$ and $\widetilde{A}$ such that $D \subseteq \mathcal{J}cl(D \cap H)$. The set of all intuitionistic i-open sets of (A, τ_I) is denoted by $\mathcal{J}iOS$.

3. Separation Axioms of $\mathcal{J}$i-open sets

3.1. $\mathcal{J}$i-T_0 space

The initiation of $\mathcal{J}$i-T_0 space is defined in this section. Also, the properties of $\mathcal{J}$i-T_0 space and the interrelationship between the space is elaborated.

Definition 3.1.1. An Intuitionistic Topological Space (A, τ_I) is said to be

(a) $T_0(i)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an intuitionistic open set G such that

$\tilde{a} \in G$ and $\tilde{b} \notin G$ or $\tilde{b} \in G$ and $\tilde{a} \notin G$.

(b) $T_0(ii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an intuitionistic open set G such

that $\tilde{\tilde{a}} \in G$, $\tilde{\tilde{b}} \notin G$ or $\tilde{\tilde{b}} \in G$, $\tilde{\tilde{a}} \notin G$.

(c) $T_0(iii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an intuitionistic open set G such that

$\tilde{a} \in G \subseteq \bar{\bar{\tilde{b}}}$ or $\tilde{b} \in G \subseteq \bar{\tilde{a}}$.

(d) $T_0(iv)$-space if for all $a, b \in K$ $(a \neq b)$ there exist an intuitionistic open set G such that

$\tilde{\tilde{a}} \in G \subseteq \bar{\bar{\tilde{\tilde{b}}}}$ or $\tilde{\tilde{b}} \in G \subseteq \bar{\tilde{\tilde{a}}}$.

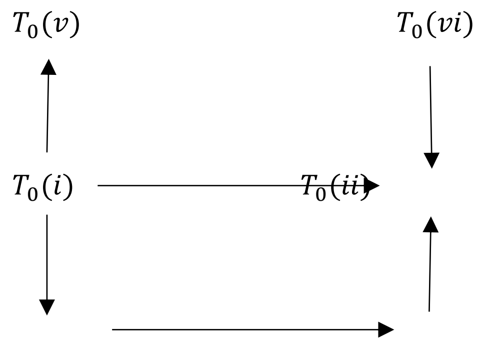
(e) $T_0(v)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an intuitionistic open set G, H such

that $\tilde{b} \notin G$ or $\tilde{a} \notin G$.

(f) $T_0(vi)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an intuitionistic open set G, H such

that $\tilde{\tilde{b}} \notin G$ or $\tilde{\tilde{a}} \notin G$.

Remark 3.1.2. The following diagram illustrate the relationship between T_0-spaces



$T_0(iii)$ $T_0(iv)$

Remark 3.1.3. The converse part of the above diagram is not true.

Example 3.1.4. Let $A = \{x, y\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, V_1\}$ where $V_1 = < A, \emptyset, \{y\} >$. Then $\tilde{\tilde{x}} = < A, \emptyset, \{y\} > \in < A, \emptyset, \{y\} > = G$. Hence $\tilde{\tilde{x}} \in G$ and $\tilde{\tilde{y}} \notin G$. Therefore (A, τ_I) is $T_0(ii)$-space. But there exist no intuitionistic open set G such that $\tilde{a} \in G$ and $\tilde{b} \notin G$ or $\tilde{b} \in G$ and $\tilde{a} \notin G$. Hence, (A, τ_I) is not a $T_0(i)$-space.

Example 3.1.5. Let $A = \{u, v\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, R_1, R_2\}$ where $R_1 = < A, \emptyset, \emptyset >$ and $R_2 = < A, \{u\}, \emptyset >$. Then $\tilde{\tilde{u}} = < A, \emptyset, \{v\} > \in < A, \emptyset, \emptyset > = G \subseteq < A, \{u\}, \emptyset >$. Hence (A, τ_I) is $T_0(iv)$-space. But $\tilde{u} = < A, \{u\}, \{v\} > \in < A, \{u\}, \emptyset > \not\subseteq < A, \{u\}, \{v\} >$. Therefore, (A, τ_I) is not a $T_0(iii)$-space.

Example 3.1.6. Let $A = \{k, l\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, E_1\}$ where $E_1 = < A, \emptyset, \emptyset >$. Then $\tilde{k} = < A, \{k\}, \{l\} > \notin < A, \emptyset, \emptyset >$. Hence (A, τ_I) is $T_0(v)$-space. But there exist no intuitionistic open set G such that $\tilde{k} \in G$ and $\tilde{l} \notin G$ or $\tilde{l} \in G$ and $\tilde{k} \notin G$. Hence, (A, τ_I) is not a $T_0(i)$-space.

Example 3.1.7. Let $A = \{m, n\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, N_1\}$ where $N_1 = < A, \{m\}, \emptyset >$. Then $\tilde{m} = < A, \{m\}, \{n\} > \in < A, \{m\}, \emptyset > = G$. Hence $\tilde{m} \in G$ and $\tilde{n} \notin G$. Therefore, (A, τ_I) is a $T_0(i)$-space. But $\tilde{m} = < A, \{m\}, \{n\} > \in < A, \{m\}, \emptyset > = G \not\subseteq \bar{\tilde{n}} =< A, \{m\}, \{n\} >$ which implies (A, τ_I) is not a $T_0(iii)$-space.

Example 3.1.8. Consider the above example Then $\tilde{\tilde{m}} = < A, \emptyset, \{n\} > \in < A, \{m\}, \emptyset > = G \subseteq \bar{\tilde{\tilde{n}}} = < A, \{m\}, \emptyset >$. Hence $\tilde{\tilde{m}} \in G \subseteq \bar{\tilde{\tilde{n}}}$. Therefore, (A, τ_I) is a $T_0(iv)$-space. But $\tilde{\tilde{g}} \in G$ and $\bar{\tilde{\tilde{f}}} \in G$ which implies (A, τ_I) is not a $T_0(ii)$-space.

Definition 3.1.9. An ITS (A, τ_I) is said to be

(a) $T_{i0}(i)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that $\tilde{a} \in G$,

$\tilde{b} \notin G$ or $\tilde{b} \in G$, $\tilde{a} \notin G$.

(b) $T_{i0}(ii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that $\tilde{\tilde{a}} \in G$,

$\tilde{\tilde{b}} \notin G$ or $\tilde{\tilde{b}} \in G$, $\tilde{\tilde{a}} \notin G$.

(c) $T_{i0}(iii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that

$\tilde{a} \in G \subseteq \bar{\tilde{b}}$ or $\tilde{b} \in G \subseteq \bar{\tilde{a}}$.

(d) $T_{i0}(iv)$-space if for all $a, b \in K$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that

$\tilde{\tilde{a}} \in G \subseteq \bar{\tilde{\tilde{b}}}$ or $\tilde{\tilde{b}} \in G \subseteq \bar{\tilde{\tilde{a}}}$.

(e) $T_{i0}(v)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that

$\tilde{b} \notin G$ or $\tilde{a} \notin G$.

(f) $T_{i0}(vi)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G such that

$\tilde{\tilde{b}} \notin G$ or $\tilde{\tilde{a}} \notin G$.

Definition 3.1.10. An intuitionistic topological space (A, τ_I) is called intuitionistic T_{i0} space
if for all $a, b \in A$ with $a \neq b$ there exist an $\mathcal{J}i$-open set G such that $a \in G_1, b \in G_2$ or $b \in G_1, a \in G_2$.

Theorem 3.1.11. Every intuitionistic T_0 space is an intuitionistic T_{i0} space .

Proof: Since every intuitionistic open is $\mathcal{J}i$-open, the proof follows.

Remark 3.1.12. The reverse implication of the above theorem is no true

Example 3.1.13. Let $A = \{k, l\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, E_1\}$ where $E_1 = < A, \emptyset, \{l\} >$. Then $l \in < A, \{l\}, \emptyset >$ and $k \notin < A, \{l\}, \emptyset >$. Hence (A, τ_I) is intuitionistic T_{i0} space. But there exist no intuitionistic open set G such that $k \in G$ and $l \notin G$ or $l \in G$ and $k \notin G$. Hence, (A, τ_I) is not an intuitionistic T_0 space.

Theorem 3.1.14. Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be a one-one, onto and $\mathcal{J}i$-open map. If (A, τ_{I_1}) is intuitionistic T_0 space then (B, τ_{I_2}) is an intuitionistic T_{i0} space.

Proof: Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be one-one, onto and $\mathcal{J}i$-open map. Let (A, τ_{I_1}) be intuitionistic T_0 space, we shall show that (B, τ_{I_2}) is an intuitionistic T_{i0} space. Suppose $a, b \in B$ with $a \neq b$. Since s is onto then there exist $p, q \in A$ such that $s(p) = a$ and $s(q) = b$. Again, since $a \neq b \Rightarrow s(p) \neq s(q) \Rightarrow p \neq q$ as s is one-one. Further since $p, q \in A, p \neq q$ and (A, τ_{I_1}) is intuitionistic T_0 space then there exist intuitionistic open set G in A such that $p \in G_1, q \notin G_1$. Since $G \in (A, \tau_{I_1}) \Rightarrow s(G) \in (B, \tau_{I_2})$ as s is $\mathcal{J}i$-open. We know, $s(G) =$ <B, $s(G_1), s(G_2) >$. Furthermore $a = s(p) \in s(G_1)$ and $b = s(q) \notin s(G_1)$. Finally, we get $a, b \in B$ with $a \neq b$ there exist an $\mathcal{J}i$-open set $s(G) \in (B, \tau_{I_2})$ such that $a = s(p) \in s(G_1)$, $b = s(q) \notin s(G_1)$ .Therefore, (B, τ_{I_2}) is an intuitionistic T_{i0} space.

Theorem 3.1.15. Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be a one-one, onto and $\mathcal{J}i$-continuous map. If (B, τ_{I_2}) is intuitionistic T_0
space then (A, τ_{I_1}) is an intuitionistic T_{i0} space.

Proof: Let $x, y \in A$ with $x \neq y$ implies $s(x), s(y) \in B$ with $s(x) \neq s(y)$ as s is one-one. Since $s(x), s(y) \in B$ and (B, τ_{I_2}) is intuitionistic T_0 space then there exist intuitionistic open set G in B such that $s(x) \in G_1$, $s(y) \notin G_1$ or $s(y) \in G_1, s(x) \notin G_1$. Now, $s(x) \in G_1$ implies $s^{-1}(s(x)) \in s^{-1}(G_1)$ which implies

$x \in s^{-1}(G_1)$. And, $s(y) \in G_1$ implies $s^{-1}(s(y)) \in s^{-1}(G_1)$ which implies $y \in s^{-1}(G_1)$. Similarly, $y \notin s^{-1}(G_1)$, $x \notin s^{-1}(G_1)$. Therefore, we get $x, y \in A$ with $x \neq y$ there exist $\mathcal{I}i$-open set $s^{-1}(G)$ such that $x \in s^{-1}(G_1)$, $y \notin s^{-1}(G_1)$ or $y \in s^{-1}(G_1)$, $x \notin s^{-1}(G_1)$. Therefore, (A, τ_{I_1}) is an intuitionistic T_{i0} space.

3.2. $\mathcal{I}i$-T_1 space

The definition, the properties and the inter relation between the $\mathcal{I}i$-T_1spaces are briefly explained in this section. Further we proved the converse part is not true with counter examples.

Definition 3.2.1. An Intuitionistic Topological Space (A, τ_I) is said to be

(a) $T_{i1}(i)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that $\tilde{a} \in G$,

$\tilde{b} \notin G$ and $\tilde{b} \in H$, $\tilde{a} \notin H$.

(b) $T_{i1}(ii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G$, $\tilde{\tilde{b}} \notin G$ and $\tilde{\tilde{b}} \in H$, $\tilde{\tilde{a}} \notin H$.

(c) $T_{i1}(iii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{a} \in G \subseteq \bar{\tilde{b}}$ and $\tilde{b} \in H \subseteq \bar{\tilde{a}}$.

(d) $T_{i1}(iv)$-space if for all $a, b \in K$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G \subseteq \bar{\tilde{\tilde{b}}}$ and $\tilde{\tilde{b}} \in H \subseteq \bar{\tilde{\tilde{a}}}$.

(e) $T_{i1}(v)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{b} \notin G$ and $\tilde{a} \notin H$.

(f) $T_{i1}(vi)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{b}} \notin G$ and $\tilde{\tilde{a}} \notin H$.

Theorem 3.2.2. Let (A, τ_I) be an ITS. Then the following implications are valid.

$$
\begin{tikzcd}
\boldsymbol{T_{i1}(v)} \arrow[r] & \boldsymbol{T_{i1}(vi)} \\
\boldsymbol{T_{i1}(i)} \arrow[u] \arrow[r] & \boldsymbol{T_{i1}(ii)} \arrow[u] \arrow[d] \\
{} \arrow[u] \arrow[r] & {}
\end{tikzcd}
$$

$\boldsymbol{T_{i1}(iii)}$ $\boldsymbol{T_{i1}(iv)}$

Proof: Trivial.

Remark 3.2.3. The reverse implication is not valid as seen from the following example.

Example 3.2.4. Let $K = \{2,3,5\}$ with $\tau_I = \{\widetilde{K}, \widetilde{\emptyset}, V_1, V_2, V_3\}$ where $V_1 = < K, \{3\}, \{2\} >, V_2 = < K, \{3\}, \{2,5\} >, V_3 = < K, \emptyset, \{2\} >$ and $V_4 = < K, \emptyset, \{2,5\} >$. Then (A, τ_I) is $T_{i1}(i)$- space but not $T_{i1}(iii)$-space.

Example 3.2.5 Let $K = \{i, j\}$ with a family $\tau_I = \{\widetilde{K}, \widetilde{\emptyset}, R_1, R_2\}$ where $R_1 = < K, \{j\}, \{i\} >$ and $R_2 = < K, \emptyset, \{i\} >$. Then (A, τ_I) is $T_{i1}(iv)$- space but not $T_{i1}(ii)$-space.

Example 3.2.6. Let $K = \{17,19,21\}$ with $\tau_I = \{\widetilde{K}, \widetilde{\emptyset}, V_1, V_2, V_3\}$ where $V_1 = < K, \{17\}, \{19\} >$, $V_2 = < K, \emptyset, \{19\} >$, $V_3 = < K, \{17,21\}, \emptyset >$ and $V_4 = < K, \{17\}, \emptyset >$. Then (A, τ_I) is $T_{i1}(vi)$-space but not $T_{i1}(ii)$-space.

Example 3.2.7. Let $K = \{\Gamma, \Delta\}$ with $\tau_I = \{\widetilde{K}, \widetilde{\emptyset}, P_1, P_2\}$ where $P_1 = < K, \emptyset, \emptyset >$ and $P_2 = < K, \{\Delta\}, \emptyset >$. Then (A, τ_I) is $T_{i1}(v)$- space but not $T_{i1}(vi)$-space.

Example 3.2.8 Let $A = \{85,91\}$ with $\tau_I = \{\tilde{A}, \widetilde{\emptyset}, Y_1, Y_2\}$ where $Y_1 = < A, \emptyset, \emptyset >$ and $Y_2 = < A, \{91\}, \emptyset >$. Then (A, τ_I) is $T_{i1}(v)$-space but not $T_{i1}(i)$-space.

Example 3.2.9. Let $A = \{i, j\}$ with $\tau_I = \left\{\tilde{A}, \widetilde{\emptyset}, H_1, H_2\right\}$ where $H_1 = < A, \{i\}, \emptyset >$ and $H_2 = < A, \emptyset, \{j\} >$. Then (A, τ_I) is $T_{i1}(iv)$-space but not $T_{i1}(iii)$-space.

Definition 3.2.10. An intuitionistic topological space (A, τ_I) is called intuitionistic T_{i1} space if for all $a, b \in A$ with $a \neq b$ there exists an $\mathcal{I}i$-open set G, H such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$.

Theorem 3.2.11. Every intuitionistic T_1 space is an intuitionistic T_{i1} space.

Proof: Since every intuitionistic open is $\mathcal{I}i$-open, the proof follows.

Remark 3.2.12. The reverse implication of the above theorem is no true

Example 3.2.13. Let $A = \{x, y\}$ with a family $\tau_I = \{\tilde{A}, \widetilde{\emptyset}, E_1\}$ where $E_1 = < A, \{y\}, \emptyset >$. Then $k \in < A, \{k\}, \emptyset >, l \notin < A, \{k\}, \emptyset >$ and $l \in < A, \{l\}, \{k\} >, k \notin < A, \{l\}, \{k\} >$. Hence (A, τ_I) is intuitionistic T_{i1}-space. But there exist no intuitionistic open set G such that $k \in G$ and $l \notin G$ and $l \in H$ and $k \notin H$. Hence, (A, τ_I) is not an intuitionistic T_1-space.

Theorem 3.2.14. Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \rightarrow (B, \tau_{I_2})$ be a one-one, onto and $\mathcal{J}i$-open map. If (A, τ_{I_1}) is intuitionistic T_1 space then (B, τ_{I_2}) is an intuitionistic T_{i1} space.

Proof: Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be one-one, onto and $\mathcal{J}i$-open map. Let (A, τ_{I_1}) be intuitionistic T_1 space, we shall show that (B, τ_{I_2}) is an intuitionistic T_{i1} space. Suppose $a, b \in B$ with $a \neq b$. Since s is onto then there exist $p, q \in A$ such that $s(p) = a$ and $s(q) = b$. Again, since $a \neq b \Rightarrow s(p) \neq s(q) \Rightarrow p \neq q$ as s is one-one. Further since $p, q \in A, p \neq q$ and (A, τ_{I_1}) is intuitionistic T_1 space then there exist intuitionistic open set G, H in A such that $p \in G_1, q \notin G_1$ and $q \in H_1, p \notin H_1$. Since $G , H \in (A, \tau_{I_1}) \Rightarrow s(G), s(H) \in (B, \tau_{I_2})$ as s is $\mathcal{J}i$-open. We know, $s(G) =$ <B, $s(G_1), s(G_2) >$ and $s(H) =$ <B, $s(H_1), s(H_2) >$. Furthermore $a = s(p) \in s(G_1)$ and $b = s(q) \in s(H_1)$. Also, $q \notin G_1$ implies $b = s(q) \notin s(G_1)$ and $p \notin H_1$ implies $a = s(p) \notin s(H_1)$. Finally, we get $a, b \in B$ with $a \neq b$ there exist an $\mathcal{J}i$-open set $s(G), s(H) \in (B, \tau_{I_2})$ such that $a = s(p) \in s(G_1)$, $b = s(q) \notin s(G_1)$ and
$b = s(q) \in s(H_1)$, $a = s(p) \notin s(H_1)$. Therefore, (B, τ_{I_2}) is an intuitionistic T_{i1} space.

Theorem 3.2.15. Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be a one-one, onto and $\mathcal{J}i$-continuous map. If (B, τ_{I_2}) is intuitionistic T_1 space then (A, τ_{I_1}) is an intuitionistic T_{i1} space.

Proof: Let $x, y \in A$ with $x \neq y$ implies $s(x), s(y) \in B$ with $s(x) \neq s(y)$ as s is one-one. Since $s(x), s(y) \in B$ and (B, τ_{I_2}) is intuitionistic T_2 space then there exist intuitionistic open set G, H in B such that $s(x) \in G_1$, $s(y) \notin G_1$ and $s(y) \in H_1, s(x) \notin H_1$. Now, $s(x) \in G_1$ implies $s^{-1}(s(x)) \in s^{-1}(G_1)$ which implies $x \in s^{-1}(G_1)$. And, $s(y) \in H_1$ implies $s^{-1}(s(y)) \in s^{-1}(H_1)$ which implies $y \in s^{-1}(H_1)$. Similarly, $y \notin s^{-1}(G_1)$, $x \notin s^{-1}(H_1)$. Finally, we get $x, y \in A$ with $x \neq y$ there exist $\mathcal{J}i$-open set $s^{-1}(G)$, $s^{-1}(H)$ such that $x \in s^{-1}(G_1)$, $y \notin s^{-1}(G_1)$ and $y \in s^{-1}(H_1)$, $x \notin s^{-1}(H_1)$. Therefore, (A, τ_{I_1}) is an intuitionistic T_{i1} space.

3.3. $\mathcal{J}i$-T_2 space

The concept of $\mathcal{J}i$-T_2 space is explained in this section. Further the properties and inter relation between the spaces is explicated with examples.

Definition 3.3.1. An Intuitionistic Topological Space (A, τ_I) is said to be

(a) $T_{i2}(i)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{J}i$-open set G, H such that $\tilde{a} \in G$,
$\tilde{b} \in H$ and $G \cap H = \tilde{\emptyset}$.

(b) $T_{i2}(ii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G, \tilde{\tilde{b}} \in H$ and $G \cap H = \tilde{\emptyset}$.

(c) $T_{i2}(iii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{a} \in G,\ \tilde{b} \in H$ and $G \subseteq \ \overline{H}$.

(d) $T_{i2}(iv)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G,\ \tilde{\tilde{b}} \in H$ and $G \subseteq \ \overline{H}$.

(e) $T_{i2}(v)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{a} \in G \subseteq \bar{\tilde{b}},\ \tilde{b} \in H \subseteq \bar{\tilde{a}}$ and $G \subseteq \ \overline{H}$.

(f) $T_{i2}(vi)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G \subseteq \bar{\tilde{\tilde{b}}}\,, \tilde{\tilde{b}} \in H \subseteq \bar{\tilde{\tilde{a}}}$ and $G \subseteq \ \overline{H}$.

(g) $T_{i2}(vii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{a} \in G, \tilde{b} \ \notin G\ , \tilde{b} \in H\ , \tilde{a}\ \notin H$ and $G \subseteq \ \overline{H}$.

(h) $T_{i2}(viii)$-space if for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that

$\tilde{\tilde{a}} \in G\ , \tilde{\tilde{b}}\ \notin G\ , \tilde{\tilde{b}} \in H\ , \tilde{\tilde{a}}\ \notin H$ and $G \subseteq \ \overline{H}$.

Theorem 3.3.2. Let (A, τ_I) be an ITS. Then the following implications are valid.

$$\begin{tikzcd}
 & \boldsymbol{T_{i2}(v)} \arrow[r] \arrow[d] & \boldsymbol{T_{i2}(vi)} \arrow[d] & \\
\boldsymbol{T_{i2}(vii)} & \boldsymbol{T_{i2}(i)} \arrow[l] \arrow[r] \arrow[d] & \boldsymbol{T_{i2}(ii)} \arrow[r] \arrow[d] & \boldsymbol{T_{i2}(viii)} \\
 & \boldsymbol{T_{i2}(iii)} \arrow[r] & \boldsymbol{T_{i2}(iv)} &
\end{tikzcd}$$

Proof: Trivial.

Remark 3.3.3. The converse of the above implications is not true

Example 3.3.4. Consider example 3.2.9. Then $\tilde{\tilde{d}} \ = <\ A\,, \emptyset, \{e\} >\ \in\ G\ = < A\,, \{d\}, \emptyset >$ and $\tilde{\tilde{e}} = <\ A\,, \emptyset, \{d\} > \in\ H\ = < A\,, \emptyset, \{d\} >$. Then $G \subseteq \ \overline{H}$. Hence, (A, τ_I) is $T_{i2}(iv)$- space but $G \cap H\ = <\ A\,, \emptyset, \{d\} > \neq\ \tilde{\emptyset}$. Hence, (A, τ_I) is not $T_{i2}(ii)$-space.

Example 3.3.5. From the above example we seen that (A, τ_I) is $T_{i2}(iv)$- space . Now $\tilde{d} =< A, \{d\}, \{e\} > \in G =< A, \{d\}, \emptyset >$ and $\tilde{e} =< A, \{e\}, \{d\} > \in H =< A, \{e\}, \emptyset >$. Then $G \nsubseteq \overline{H}$. Hence (A, τ_I) is not $T_{i2}(iii)$-space.

Example 3.3.6. Let $A = \{k, l\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, < A, \{k\}, \emptyset >, < A, \emptyset, \{l\} >, < A, \{k\}, \{l\} >\}$. Then $\tilde{\tilde{k}} =< A, \emptyset, \{l\} > \in G =< A, \emptyset, \{l\} > \subseteq < A, \{k\}, \emptyset >$ and $\tilde{\tilde{l}} =< A, \emptyset, \{k\} > \in H =< A, \emptyset, \{k\} > \subseteq < A, \{l\}, \emptyset >$. Also, $G \subseteq \overline{H}$. Hence, (A, τ_I) is $T_{i2}(vi)$-space. Now, $\tilde{k} =< A, \{k\}, \{l\} > \in G =< A, \{k\}, \{l\} >$ and $\tilde{l} =< A, \{l\}, \{k\} > \in H =< A, \{l\}, \emptyset >$. But, $H \nsubseteq \overline{\tilde{k}}$. Hence, (A, τ_I) is not $T_{i2}(v)$-space.

Example 3.3.7. Let $A = \{f, g, h\}$ with $\tau_I = \{\tilde{A}, \tilde{\emptyset}, < A, \{f\}, \{h\} >, < A, \emptyset, \{h\} >, < A, \{g\}, \{h\} >, < A, \{f, g\}, \{h\} >\}$. Now $\tilde{f} =< A, \{f\}, \{g, h\} > \in < A, \{f\}, \{g, h\} >$, $\tilde{g} =< A, \{g\}, \{f, h\} > \in < A, \{g\}, \{f, h\} >$ and $\tilde{h} =< A, \{h\}, \{f, g\} > \in < A, \{h\}, \{f\} >$. Then for all $f, g \in A$ $(f \neq g)$ there exist an $\mathcal{I}i$-open set G, H such that $\tilde{f} \in G$, $\tilde{g} \notin G$, $\tilde{g} \in H$, $\tilde{f} \notin H$ and $G \subseteq \overline{H}$. Hence, (A, τ_I) is $T_{i2}(vii)$-space. But $G \cap H =< A, \{g\}, \{f, h\} > \cap < A, \{h\}, \{f\} > =< A, \emptyset, \{f, h\} > \neq \tilde{\emptyset}$. Hence, (A, τ_I) is not $T_{i2}(i)$-space.

Example 3.3.8. Let $A = \{f, g, h\}$ with $\tau_I = \{\tilde{A}, \tilde{\emptyset}, < A, \{f\}, \{g, h\} >, < A, \{g, h\}, \{f\} >\}$. Now $\tilde{\tilde{f}} =< A, \emptyset, \{g, h\} > \in < A, \emptyset, \{g, h\} >$, $\tilde{\tilde{g}} =< A, \emptyset, \{f, h\} > \in < A, \emptyset, \{f, h\} >$ and $\tilde{\tilde{h}} =< A, \emptyset, \{f, g\} > \in < A, \emptyset, \{f, g\} >$. Then for all $a, b \in A$ $(a \neq b)$ there exist an $\mathcal{I}i$-open set G, H such that $\tilde{\tilde{a}} \in G$, $\tilde{\tilde{b}} \notin G$, $\tilde{\tilde{b}} \in H$, $\tilde{\tilde{a}} \notin H$ and $G \subseteq \overline{H}$. Hence, (A, τ_I) is $T_{i2}(viii)$-space. But $G \cap H \neq \tilde{\emptyset}$ for all $a, b \in A$ $(a \neq b)$. Hence, (A, τ_I) is not $T_{i2}(ii)$-space.

Definition 3.3.9. An intuitionistic topological space (A, τ_I) is called intuitionistic T_{i2} space if for all $a, b \in A$ with $a \neq b$ there exists an $\mathcal{I}i$-open set G, H such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = \tilde{\emptyset}$.

Theorem 3.3.10. Every intuitionistic T_2 space is an intuitionistic T_{i2} space .

Proof: Since every intuitionistic open is $\mathcal{I}i$-open, the proof follows.

Remark 3.3.11. The reverse implication of the above theorem is no true

Example 3.3.12. Let $A = \{k, l\}$ with a family $\tau_I = \{\tilde{A}, \tilde{\emptyset}, < A, \emptyset, \emptyset >\}$. Then $k \in G =< A, \{k\}, \{l\} >$, $l \notin G$ and $l \in H =< A, \{l\}, \{k\} >$, $k \notin H$. Also, $G \cap H = \tilde{\emptyset}$. Hence, (A, τ_I) is an intuitionistic T_{i2} space. But there exist no intuitionistic open set G, H such that $a \in G_1, b \notin G_1$ and $b \in H_1, a \notin H_1$ and $G \cap H = \tilde{\emptyset}$.

Theorem 3.3.13. Let (A, τ_{I_1}) and (B, τ_{I_2}) be two intuitionistic topological spaces and $s : (A, \tau_{I_1}) \to (B, \tau_{I_2})$ be a one-one, onto and $\mathcal{J}i$-open map. If (A, τ_{I_1}) is intuitionistic T_2 space then (B, τ_{I_2}) is an intuitionistic $\mathrm{T_{i2}}$ space.

Proof: Let $(\mathrm{A}, \tau_{\mathrm{I_1}})$ and $(\mathrm{B}, \tau_{\mathrm{I_2}})$ be two intuitionistic topological spaces and $\mathrm{s} : (\mathrm{A}, \tau_{\mathrm{I_1}}) \to (\mathrm{B}, \tau_{\mathrm{I_2}})$ be one-one, onto and $\mathcal{J}$i-open map. Let $(\mathrm{A}, \tau_{\mathrm{I_1}})$ be intuitionistic $\mathrm{T_2}$ space, we shall show that $(\mathrm{B}, \tau_{\mathrm{I_2}})$ is an intuitionistic $\mathrm{T_{i1}}$ space. Suppose $\mathrm{a}, \mathrm{b} \in \mathrm{B}$ with $\mathrm{a} \neq \mathrm{b}$. Since s is onto then there exist $\mathrm{p}, \mathrm{q} \in \mathrm{A}$ such that $\mathrm{s(p)} = \mathrm{a}$ and $\mathrm{s(q)} = \mathrm{b}$. Again, since $\mathrm{a} \neq \mathrm{b} \Rightarrow \mathrm{s(p)} \neq \mathrm{s(q)} \Rightarrow \mathrm{p} \neq \mathrm{q}$ as s is one-one. Further since $\mathrm{p}, \mathrm{q} \in \mathrm{A}, \mathrm{p} \neq \mathrm{q}$ and $(\mathrm{A}, \tau_{\mathrm{I_1}})$ is intuitionistic $\mathrm{T_2}$ space then there exist intuitionistic open set G, H in A such that $\mathrm{p} \in \mathrm{G_1}, \mathrm{q} \notin \mathrm{G_1}$ and $\mathrm{q} \in \mathrm{H_1}, \mathrm{p} \notin \mathrm{H_1}$ and $\mathrm{G} \cap \mathrm{H} = \widetilde{\emptyset}$. Since $\mathrm{G}, \mathrm{H} \in (\mathrm{A}, \tau_{\mathrm{I_1}}) \Rightarrow \mathrm{s(G)}, \mathrm{s(H)} \in (\mathrm{B}, \tau_{\mathrm{I_2}})$ as s is $\mathcal{J}$i-open. We know, $\mathrm{s(G)} = <\mathrm{B}, \mathrm{s(G_1)}, \mathrm{s(G_2)} >$ and $\mathrm{s(H)} = <\mathrm{B}, \mathrm{s(H_1)}, \mathrm{s(H_2)} >$. Furthermore $\mathrm{a} = \mathrm{s(p)} \in \mathrm{s(G_1)}$ and $\mathrm{b} = \mathrm{s(q)} \in \mathrm{s(H_1)}$. Also, $\mathrm{q} \notin \mathrm{G_1}$ implies $\mathrm{b} = \mathrm{s(q)} \notin \mathrm{s(G_1)}$ and $\mathrm{p} \notin \mathrm{H_1}$ implies $\mathrm{a} = \mathrm{s(p)} \notin \mathrm{s(H_1)}$. Consider $\mathrm{s(G)} \cap \mathrm{s(H)} \neq \widetilde{\emptyset}$ which implies $\mathrm{s(G_1)} \cap \mathrm{s(H_1)} \neq \emptyset$ then there exists at least one $\mathrm{c} \in \mathrm{B}$ for which $\mathrm{c} \in \mathrm{s(G_1)} \cap \mathrm{s(H_1)}$ which implies $\mathrm{c} \in \mathrm{s(G_1)}$ and $\mathrm{c} \in \mathrm{s(H_1)}$. Then, there exists $\mathrm{u} \in \mathrm{G_1}$ and $\mathrm{v} \in \mathrm{H_1}$ such that $\mathrm{s(u)} = \mathrm{s(v)} = \mathrm{c} \Rightarrow \mathrm{u} = \mathrm{v}$ as s is one-one $\Rightarrow \mathrm{u} = \mathrm{v} \in \mathrm{G_1} \cap \mathrm{H_1}$ which is a contradiction to the fact that $\mathrm{G} \cap \mathrm{H} = \widetilde{\emptyset}$. Therefore, we get $\mathrm{s(G)} \cap \mathrm{s(H)} = \widetilde{\emptyset}$. Finally, we get $\mathrm{a}, \mathrm{b} \in \mathrm{B}$ with $\mathrm{a} \neq \mathrm{b}$ there exist an $\mathcal{J}$i-open set $\mathrm{s(G)}, \mathrm{s(H)} \in (\mathrm{B}, \tau_{\mathrm{I_2}})$ such that $\mathrm{a} = \mathrm{s(p)} \in \mathrm{s(G_1)}$, $\mathrm{b} = \mathrm{s(q)} \notin \mathrm{s(G_1)}$ and $\mathrm{b} = \mathrm{s(q)} \in \mathrm{s(H_1)}$, $\mathrm{a} = \mathrm{s(p)} \notin \mathrm{s(H_1)}$ and $\mathrm{s(G)} \cap \mathrm{s(H)} = \widetilde{\emptyset}$. Therefore, $(\mathrm{B}, \tau_{\mathrm{I_2}})$ is an intuitionistic $\mathrm{T_{i2}}$ space.

Theorem 3.3.14. Let $(\mathrm{A}, \tau_{\mathrm{I_1}})$ and $(\mathrm{B}, \tau_{\mathrm{I_2}})$ be two intuitionistic topological spaces and $\mathrm{s} : (\mathrm{A}, \tau_{\mathrm{I_1}}) \to (\mathrm{B}, \tau_{\mathrm{I_2}})$ be a one-one, onto and $\mathcal{J}$i-continuous map. If $(\mathrm{B}, \tau_{\mathrm{I_2}})$ is intuitionistic $\mathrm{T_2}$ space then $(\mathrm{A}, \tau_{\mathrm{I_1}})$ is an intuitionistic $\mathrm{T_{i2}}$ space.

Proof: Let $\mathrm{x}, \mathrm{y} \in \mathrm{A}$ with $\mathrm{x} \neq \mathrm{y}$ implies $\mathrm{s(x)}, \mathrm{s(y)} \in \mathrm{B}$ with $\mathrm{s(x)} \neq \mathrm{s(y)}$ as s is one-one. Since $\mathrm{s(x)}, \mathrm{s(y)} \in \mathrm{B}$ and $(\mathrm{B}, \tau_{\mathrm{I_2}})$ is intuitionistic $\mathrm{T_2}$ space then there exist intuitionistic open set G, H in B such that $\mathrm{s(x)} \in \mathrm{G_1}$, $\mathrm{s(y)} \notin \mathrm{G_1}$ and $\mathrm{s(y)} \in \mathrm{H_1}, \mathrm{s(x)} \notin \mathrm{H_1}$ and $\mathrm{G} \cap \mathrm{H} = \widetilde{\emptyset}$ which implies $\mathrm{G_1} \cap \mathrm{H_1} = \emptyset$. Now, $\mathrm{s(x)} \in \mathrm{G_1}$ implies $\mathrm{s^{-1}(s(x))} \in \mathrm{s^{-1}(G_1)}$ which implies $\mathrm{x} \in \mathrm{s^{-1}(G_1)}$. And, $\mathrm{s(y)} \in \mathrm{H_1}$ implies $\mathrm{s^{-1}(s(y))} \in \mathrm{s^{-1}(H_1)}$ which implies $\mathrm{y} \in \mathrm{s^{-1}(H_1)}$. Similarly, $\mathrm{y} \notin \mathrm{s^{-1}(G_1)}$, $\mathrm{x} \notin \mathrm{s^{-1}(H_1)}$. Suppose $\mathrm{s^{-1}(G)} \cap \mathrm{s^{-1}(H)} \neq \widetilde{\emptyset}$ which implies $\mathrm{s^{-1}(G_1)} \cap \mathrm{s^{-1}(H_1)} \neq \emptyset$ which implies $\mathrm{s(s^{-1}(G_1))} \cap \mathrm{s(s^{-1}(H_1))} \neq \emptyset$ which implies $\mathrm{G_1} \cap \mathrm{H_1} \neq \emptyset$ which is a contradiction. Therefore,

$s^{-1}(G) \cap s^{-1}(H) = \widetilde{\emptyset}$. Finally, we get $x, y \in A$ with $x \neq y$ there exist $\mathcal{J}$i-open set $s^{-1}(G)$, $s^{-1}(H)$ such that $x \in s^{-1}(G_1)$, $y \notin s^{-1}(G_1)$ and $y \in s^{-1}(H_1)$, $x \notin s^{-1}(H_1)$. and $s^{-1}(G) \cap s^{-1}(H) = \widetilde{\emptyset}$. Therefore, (A, τ_{I_1}) is an intuitionistic T_{i2} space.

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