Transmuted Hazard Constant Hazard Rate Model

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Abstract

In this study , a new Transmuted Hazard Constant Formula (THCF) is discovered by combining the transmuted survival formula with the constant hazard function. Because a constant hazard function is considered important and necessary in the field of biostatistics and analyzing survival data, this new formula (THF) yields new distributions whose results in analyzing life data are superior to those of the original life distribution.

Keywords : the constant hazard function , survival function , transmuted formula, transmuted survival formula ,density function .

Introduction

It is noticeable that life differs in its assets, whether it is a living organism (such as a person or an animal), smart devices (such as a mobile phone or a computer), as well as human gatherings (such as an organization or workforce) or the assembly of a group of devices (such as a communication network or a group of machines) that have a specific survival period. He may get sick, and therefore it may be a severe disease that leads to death, or it may be mild and life continues, as well as the labor forces may stop due to injury, and the same thing with regard to machines may break down or continue to work. The danger is present and possible for the organism and the machine as well. The risk function is of great importance in biostatistics because it is used to estimate risk rates over the years.it noted in the research [1] that the researcher touched on the idea that the risk function can be used and benefited from in survival . In the research [2], the researchers noted that the constant hazard function is The gravity function of the exponential distribution is an exponential distribution.in the papers [3] and [4], the researchers dealt with the concept of life survival at the time of the event and the time of the occurrence of the event. The researchers found in the study [5], the interpretation of the risk and the procedures for its occurrence are important to reduce the disaster. In the research [6], the researchers found The study of the hazard function is a key point in the survival analysis. In the study [7].

1. Basic definitions.

1.1. the hazard function.[8]

the hazard function $h(t)$ which is defined by

$$h(t) = \lim_{\Delta t \to 0} \frac{pro(t \le T < t + \Delta t | T \ge t, Y = y)}{\Delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

1.2. The survival function $S(t)$ [9].

$$S(t) = e^{-\int_0^t h(y)dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

1.3. The transmuted survival formula $S_2(t)$ [10].

$$S_2(t) = (1+\lambda)S^2(t) - \lambda S(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

1.4 IF The hazard function $h(t)$ is constant

$$h(t) = a \qquad \forall t > 0, a > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The Transmuted Constant Hazard Formula (TCH) .

Theorem(1) The new formula, known as the transmuted constant hazard formula TCH is

$$h_2(t) = a\left[\frac{\lambda}{[(1+\lambda)\, e^{-at} - \lambda]} + 2 \right] \ldots\ldots\ldots\ldots(5)$$

Proof

the new formula also depends on the following three equations (4),(2) and (3).To calculate the new formula for the hazard function , we will substitute equation (2) in to the formula (3).

$$e^{-\int_0^t h_2(y)dy} = e^{-2\int_0^t h(y)dy} + \lambda\, e^{-2\int_0^t h(y)dy} - \lambda\, e^{-\int_0^t h(y)dy}$$

Again

$$e^{-\int_0^t h_2(y)dy} = e^{-\int_0^t h(y)dy} \left[\left(e^{-2\int_0^t h(y)dy} + \lambda\, e^{-2\int_0^t h(x)dx}\right) - \lambda\right]$$

Taking the logarithm to both sides .

$$\ln e^{-\int_0^t h_2(y)dy} = \ln\left(e^{-\int_0^t h(y)dy} \left[\left(e^{-\int_0^t h(y)dy} + \lambda\, e^{-\int_0^t h(y)dy}\right) - \lambda\right]\right)$$

Then

$$-\int_0^t h_2(y)dy = -\int_0^t h(y)dy + \ln\left[e^{-\int_0^t h(y)dy} + \lambda\left(e^{-\int_0^t h(y)dy} - 1\right)\right]$$

By deriving both sides

$$-h_2(t) = -h(t) - \frac{(1+\lambda)h(t)\, e^{-\int_0^t h(x)dx}}{\left[(1+\lambda)\, e^{-\int_0^t h(x)dx} - \lambda\right]}$$

Some illustrative steps.

$$h_2(t) = h(t)\left[1 + \frac{(1+\lambda)\, e^{-\int_0^t h(x)dx}}{\left[(1+\lambda)\, e^{-\int_0^t h(x)dx} - \lambda\right]}\right]$$

$$h_2(t) = h(t)\left[1 + \frac{(1+\lambda)\, e^{-\int_0^t h(x)dx} - \lambda + \lambda}{\left[(1+\lambda)\, e^{-\int_0^t h(x)dx} - \lambda\right]}\right]$$

From equation (1.4)

$$h_2(t) = a\left[\frac{\lambda}{[(1+\lambda)\, e^{-at} - \lambda]} + 2\right]$$

$$h_2(t) = h(t)\left[\frac{(1+\lambda)\, e^{-at}}{[(1+\lambda)\, e^{-at} - \lambda]} + 1\right]$$

Therefore, this is the new formula, known as the transmuted hazard formula TCH , that is discussed further down:

$$h_2(t) = h(t)\left[\frac{(1+\lambda)\, e^{-\int_0^t h(y)dy} - \lambda + \lambda}{\left[(1+\lambda)\, e^{-\int_0^t h(y)dy} - \lambda\right]} + 1\right]$$

Simplifying the previous formula $h_2(t) = a\left[\frac{\lambda}{[(1+\lambda)\, e^{-a} - \lambda]} + 2\right]$.

Theorem(2) The function $h_2(t)$ is a hazard function

Proof

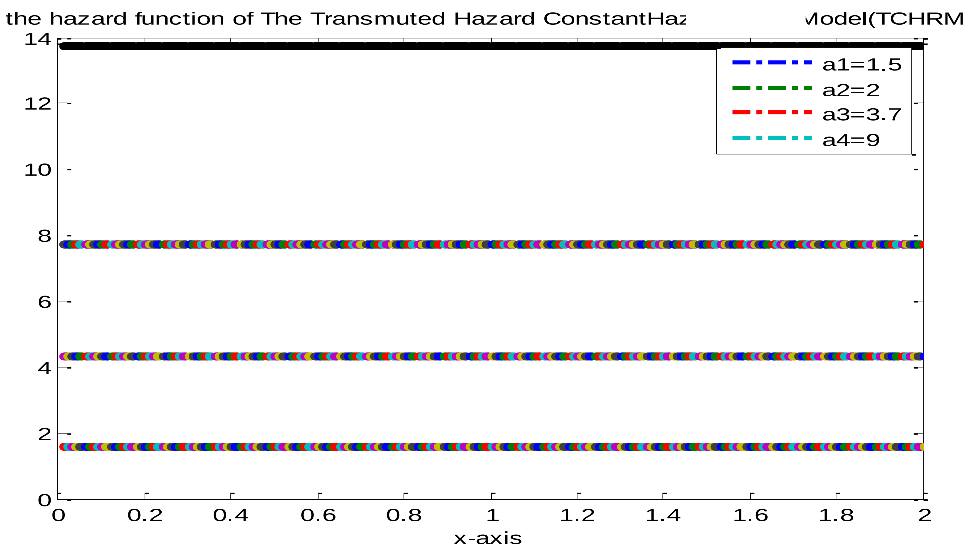
the following conditions of hazard function

satisfy the conditions of the hazard function as follows .

1. $h_2(t) \geq 0 \quad \forall \ t \geq 0$ and for all $\lambda \geq 2(1 - 2\, e^{-at})^{-1}\, e^{-at}$.

2. $\int_0^{\infty} h_2(t)\, dt = \left[\int_0^{\infty} \frac{a(1+\lambda)\, e^{-at}}{[(1+\lambda)\, e^{-at}-\lambda]}\, dt \quad + \int_0^{\infty} a dt\right]$

$$\int_0^{\infty} h_2(t)\, dt = [\ln(-\lambda) - \ln(1) \quad + \infty] = \infty$$



Figure(1): The hazard function of THCF , where the parameter a take different values .

It's been brought to our attention that The hazard function of THCHRM is monotone in the sense that the risk of harm grows with the unit's age, but reduces when its condition improves.

The Density Function of THCHRM .

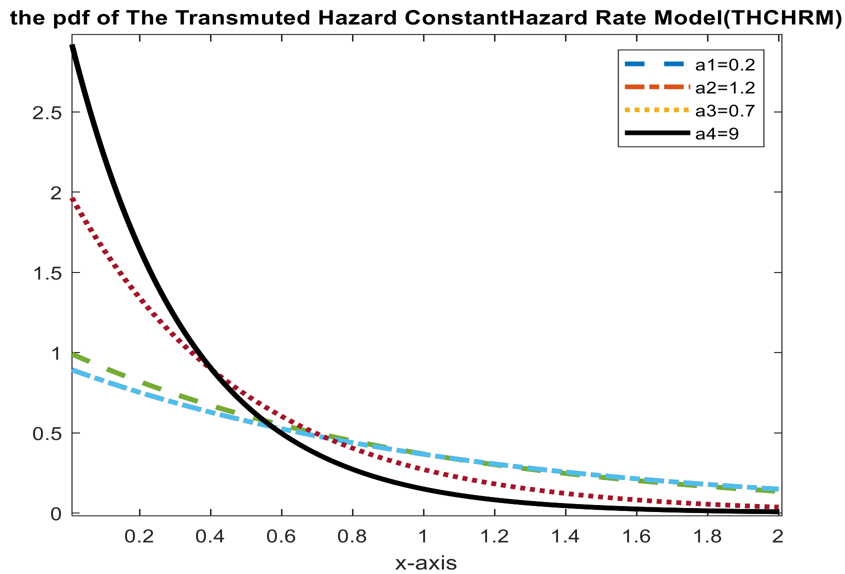
Theorem (3) The Density Function of THCHRM is $f_2(t) = a\, e^{-at}[2(1+\lambda)\, e^{-at} - \lambda\,]$ … …………….(6)

Proof :

Check the density function conditions

$1 - f_2(t) = a\, e^{-at}[2(1+\lambda)\, e^{-at} - \lambda\,] \geq 0$

$2 - \int_0^{\infty} f_2(t)dt = \int_0^{\infty} a\, e^{-at}[2(1+\lambda)\, e^{-at} - \lambda\,]dt = 1$



Figure(5.2): The Density Function of THCHRM, where the parameter a take different values . Notice that the function is increasing and then decreasing as you get closer $t \to \infty$.

Remark(7)

Another technique to show that $h_2(t)$ meets the requirements of the hazard function, using the density function and survival function as evidence.

1) $f_2\,(t\,) = a\,e^{-at}[2(1+\lambda)\,e^{-at} - \lambda\,] \geq 0$;

And $S_2(t) = (1+\lambda)S^2(t) - \lambda S(t) > 0\ thus$

$$h_2(t) = \frac{f_2(t)}{s_2(t)} \geq 0\,.$$

2) $\int_0^\infty h_2\,(t\,)dt = -\int_0^\infty dlns_2\,(t\,) = -lns_2\,(t\,)|_0^\infty = lns_2\,(0\,) - lns_2\,(\infty\,) = ln1 - ln0 = \infty$

3) $h_2\,(t\,) \geq 0$ and $\lim_{t\to 0} f_2\,(t\,) = \lim_{t\to 0} h_2\,(t\,)\,; f_2\,(t\,) \geq h_2\,(t\,)\,\forall\,t > 0$

To prove this

$$\lim_{t\to 0} f_2\,(t\,) = a\lim_{t\to 0}\ e^{-at}[2(1+\lambda)\,e^{-at} - \lambda\,] = a(2+\lambda).$$

$$\lim_{t\to 0} h_2(t) \ = a\lim_{t\to 0}\left[\ \frac{\lambda}{[(1+\lambda)\,e^{-at} - \lambda]} + 2\right] = a(2+\lambda).$$

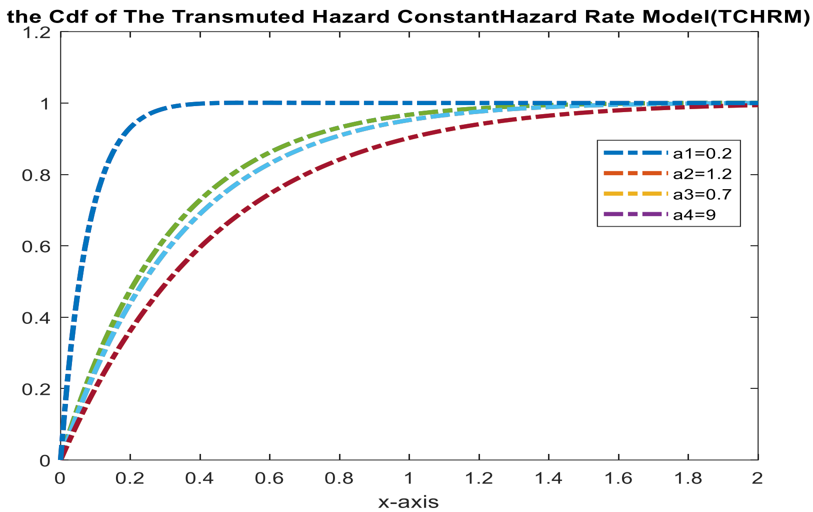
And

since it's $f_2(t) = h_2(t)\,e^{-\int_0^t h_2(y)dy}$ and $e^{-\int_0^t h_2(y)dy} \geq 0$

then $f_2(t) \geq h_2(t)$.

The Cumulative Function of THCHRM .

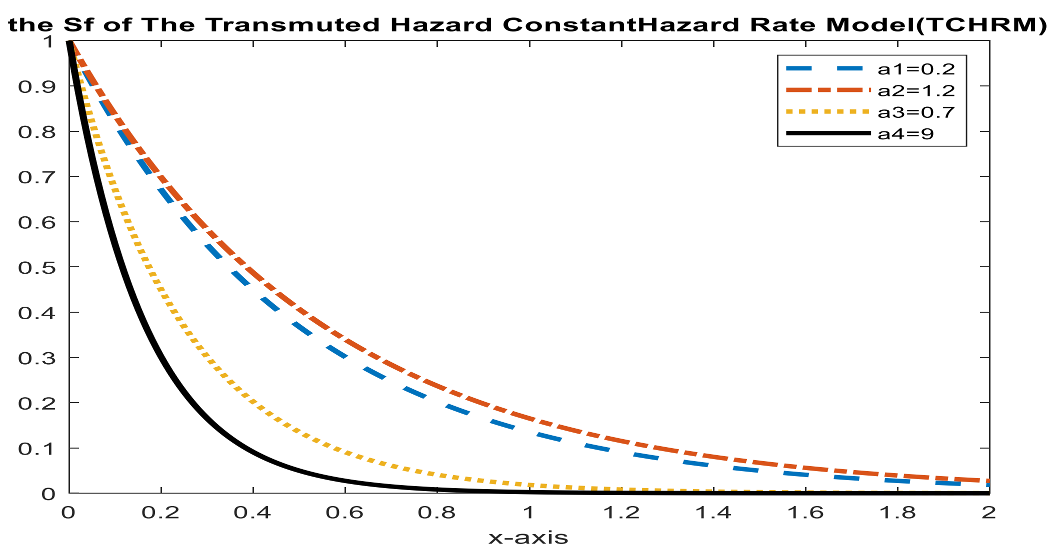
$$F_2(t) = 1 - e^{-\int_0^t h_2(t)dy} = 1 - \left((1+\lambda)e^{-at} - \lambda\right) \quad e^{-at} \text{ ....(8)}$$



Figure(5.3): The Cumulative Function of THCHRM, where the parameter λ takes number 1 . This function begins at zero and steadily increases as it moves closer and closer to the value of one when $t \to \infty$.

The Survival Function of THCHRM .

$$S_2(t) = e^{-\int_0^t h_2(t)dy} = \quad e^{-\lambda a \int_0^t \frac{}{\left((1+\lambda)e^{-at} - \lambda\right)} dy} \quad e^{-2at} \text{ ....(9)}$$



Figure(5.4): The Survival Function of THCHRM, where the parameter a take different values and $\lambda = -1$. This function starts at one and continues decreasing until it approaches zero where $t \to \infty$.

2.1 Shape of THCHRM .

2.1.1 The Limit of cumulative function of THC .

$$\lim_{t\to 0} F_2(t) = \lim_{t\to 0} \quad 1 - \left((1+\lambda)e^{-at} - \lambda\right) \quad e^{-at}$$

$$\lim_{t\to 0} F_2(t) = 1 - \lim_{t\to 0} \left((1+\lambda)e^{-at} - \lambda\right) \lim_{t\to 0} e^{-at} = 1 - 1 = 0$$

And

$$\lim_{t\to \infty} F_2(t) = 1 - \lim_{t\to \infty} \quad \left((1+\lambda)e^{-at} - \lambda\right) \quad e^{-at}$$

$$\lim_{t\to \infty} F_2(t) = 1 - \lim_{t\to \infty} \left((1+\lambda)e^{-at} - \lambda\right) \lim_{t\to \infty} e^{-at} = 1$$

2.1.2 The Limit of survival function of THCHRM .

$$\lim_{t\to 0} S_2(t) = \lim_{t\to 0} \left((1+\lambda)e^{-at} - \lambda\right) \quad e^{-at} = 1$$

And $\lim_{t\to \infty} S_2(t) = 0$

2.2 Moment THCHRM.

Theorem(5.3)
If $T\sim \text{THCHRM } (a)$ then the r_{th}moment about the origin , and about the mean μ one help hey as of T, say μ'_r , is given as

1. $E_2(T^r) = \left[\frac{2(1+\lambda)a}{(2a+1)^{r+1}} r! \quad - \frac{\lambda}{(a)^{r+1}} \; r! \right]$..............................(10)
2. $E_2(T^r) = \sum_{k=0}^{r} C_k^r \left[\frac{2(1+\lambda)a}{(2a+1)^{k+1}} k! \quad - \frac{\lambda}{(a)^{k+1}} \; k! \right] \left[\frac{2(1+\lambda)a}{(2a+1)^2} - \frac{\lambda}{a^2} \right]^{r-k}$(11)

Proof

Proof one part from theorem

Take $E_2(T^r) = \int_0^\infty t^r \, \theta \, e^{-\theta t}\left[2(1+\lambda)\, e^{-\theta t} - \lambda\,\right]dt$

$$E_2(T^r) = \left[2(1+\lambda)\theta \int_0^\infty t^r \; e^{-(2\theta+1)t}dt - \lambda\theta \int_0^\infty t^r \; e^{-\theta t}dt\right]$$

Suppose that $L_1 = \int_0^\infty t^r \; e^{-\theta t}dt$ and $L_2 = \int_0^\infty t^r \; e^{-(2\theta+1)t}dt$ and

Let $y = \theta t$ than $dy = \theta dt$

$$L_1 = \int_0^{\infty} (\frac{y}{\theta})^r \quad e^{-y} \frac{dy}{\theta}$$

$$L_1 = \left(\frac{1}{\theta}\right)^{r+1} \quad \Gamma(r+1)$$

Same that $L_2 = \int_0^{\infty} t^r \ e^{-(2\theta+1)t} dt$

Let $z = (2\theta + 1)\, t$, $dz = (2\theta + 1) dt$

$$L_2 = (\frac{1}{2\theta+1})^{r+1} \int_0^{\infty} (z)^r \quad e^{-z} dz$$

$$L_2 = (\frac{1}{2\theta+1})^{r+1}$$

As a result,

$$E_2(T^r) = \left[\frac{2(1+\lambda)a}{(2a+1)^{r+1}} r! \quad - \frac{\lambda}{(a)^{r+1}} r! \right]$$

Proof two part

Depending on the equations (10) and Binomial theorem.

When $E_2(T\ ) = \frac{2(1+\lambda)a}{(2a+1)^2} \quad - \frac{\lambda}{a^2}$.

Then

$$E_2(T^r) = \sum_{k=0}^{r} C_k^r \left[\frac{2(1+\lambda)a}{(2a+1)^{k+1}} k! \quad - \frac{\lambda}{(a)^k} k! \right] \left[\frac{2(1+\lambda)a}{(2a+1)^2} \quad - \frac{\lambda}{a} \right]^{r-k}.$$

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